A CONSTRUCTION OF MULTIDIMENSIONAL REFLECTED h-BROWNIAN MOTION

By

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To my Parents, my Teachers

and

my Friends

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Since Robert Brown, an English botanist, observed in 1828 the highly irregular motion of pollen grains suspended in water, Brownian motion has been an object of study in both pure and applied mathematics. From a probabilistic standpoint, it is viewed as the most basic stochastic process with continuous time parameter. Its application is involved in mathematical models of random phenomena arising in the natural environment. From an analytic standpoint, Brownian motion is used to develop powerful tools in functional analysis, due to its close connection with Wiener measure and Newtonian potential theory.

As a special example of a diffusion process, Brownian motion is studied for its behavior in a variety of state spaces with particular boundary conditions. Our interest in this study is the path-by-path construction of a diffusion process from Brownian motion in $\mathbb{R}^d$ where $d \geq 3$.

The construction is based on inversions with respect to spheres in $\mathbb{R}^d$. We investigate certain geometric properties. A systematic method is developed to partition the state space into a series of closed subsets. A superharmonic function h is defined on $\mathbb{R}^d$ which has an invariance property under inversions. Brownian motion is then transformed, in both time and space, while its fundamental properties as a diffusion is retained.

The resulting diffusion process has normal reflecting boundaries. In the interior of its state space, it is an h-Brownian motion.

CHAPTER 1
INTRODUCTION

1.1 Overview

A diffusion process is a strong Markov process with continuous sample paths. In general, a diffusion process can be constructed by analytic methods or probabilistic methods.

The analytic methods center on the solution of a parabolic partial differential equation with certain coefficients. The major difficulty with this approach is that it is generally a non-trivial problem to prove that this parabolic equation has a sufficiently large family of solutions with required properties to characterize the associated diffusion process. Besides, since the knowledge about this process can only be obtained by studying the associated parabolic equation, there is a loss of probabilistic intuition.

Another approach is to construct the diffusion process path-by-path, on the basis of an underlying probability space $(\Omega, \mathcal{F}, P)$. More probabilistic insights can be gained by this approach since the process is studied directly in its state space. One way to achieve this is by Itô's method of stochastic integral equations. Another way is to construct functions of well studied Markov processes, such as Brownian motion. This is the method used in this thesis.

Specifically, starting from Brownian motion, we construct a multi-dimensional diffusion process which is an h-Brownian motion with normal reflecting boundary. The state space of this process is the complement of n separated open unit balls in $\mathbb{R}^d$, $d \geq 3$.

The topic of Brownian motion, with reflecting boundary has been studied with both probabilistic and analytic approaches. Results on one-dimensional reflected Brownian motion can be found in Hida [1] and Dynkin [2]. In Chung and Williams [3], they discussed the difficulty in the study of reflecting Brownian motion in two or more dimensions. Analytical theory associated with these processes and references to earlier articles can be found in Harrison and Reiman [4].

Glover and Rao [5] studied reflected Brownian motion in $\mathbb{R}^2$ for a special class of domains using inversions. In that article, they developed a systematic partition of $\mathbb{R}^2$. A Markov process is then constructed path-by-path though time-changed Brownian motion. Finally, this process is identified as Brownian motion with normal reflection on the boundary of its state space.

Their work is the motivation of this study. One significant difference between $\mathbb{R}^2$ and $\mathbb{R}^d$, $d \geq 3$ is that Brownian motion is recurrent in the former but transient in the latter. This difference implies that no reflected Brownian motion can be constructed in $\mathbb{R}^d, d \geq 3$ by an appropriate choice of time change and inversions. However, under certain constraints, a special superharmonic function h can be defined on a complement of a polar set in $\mathbb{R}^d$ such that, for an appropriate time change, we can construct, path-by-path, an h-Brownian motion with normal reflection on its state space.

This construction is the main result of this thesis.

1.2 Critical Components

The critical components of this path-by-path construction pose several problems that have not yet been sufficiently addressed in existing literature. These problems have to be solved in order to complete our construction. We are going to highlight these critical problems in this section and start rigorous definitions and proofs in Chapter 2.

For each open ball $B_j = B(c_j, 1), j = 1, 2, \ldots, n$, the inversion with respect to the sphere ∂B_j plays an important part in the construction. It is defined as

$$I_j(x) = c_j + \frac{1}{\|x - c_j\|^2}(x - c_j)$$

for $x \in \mathbb{R}^d - \{c_j\}$. Geometrically, it maps a point $x \in B_j$ to a point lying on the ray starting at c_j and passing through x, such that

$$\|I_j(x) - c_j\| \times \|x - c_j\| = 1$$

Obviously $I_j(x) \notin B_j$. If $x \notin B_j$, then $I_j(x) \in B_j$ and the same identity holds. This collection of inversions generates a non-Abelian group $\mathcal{G}$ of homeomorphisms.

The first problem is to understand the properties of the composition of these inversions. Although inversion with respect to a single sphere has been introduced and studied in the literature, the geometric properties of inversions with respect to n spheres, where $n > 1$, have not been given detailed treatment. It is necessary to fully investigate these properties, which are not always trivial due to the non-Abelian nature of the group $\mathcal{G}$.

The second problem is also associated with the non-Abelian nature of $\mathcal{G}$. In $\mathbb{R}^2$, Brownian motion is recurrent and an appropriate time change will result in a Markov process that is invariant under inversions in $\mathcal{G}$. However, since Brownian motion is transient in $\mathbb{R}^d$ where $d \geq 3$, a time change alone is no longer sufficient to guarantee this invariant property. An h-transform is therefore necessary to be coupled with the time change. This requires that the superharmonic function h has to satisfy the identity

$$h(x) = \frac{1}{\|x - c_j\|^{d-2}} h(I_j(x))$$

for every $1 \leq j \leq n$. Although it is easy to find a superhamonic function that satisfies this condition in the case of $n = 1$, the existence of such an h, even in the case of $n = 2$, is not obvious. For any intuitive choices of the superharmonic function, the requirement of one inversion is not compatible with that of another. The difficulty here is to find one superharmonic function h that simultaneously satisfies the above identity for all $1 \leq j \leq n$.

The next problem involves whether a many-to-one continuous function maps a Markov process to a Markov process. As is pointed out in Glover and Mitro [6], this is not always true. The particular continuous function we employ has to be studied to insure that what we construct is indeed a Markov process.

Finally, since this diffusion is constructed path-by-path from a Brownian motion, its distribution can only be studied through that of the Brownian motion. We have to characterize the semigroup of this diffusion by computing its infinitesimal generator based on its connection with the original Brownian motion.

1.3 Outline of the Construction

Chapter 2 covers the study of the geometric properties of inversions and the state space. Definitions and basic properties are discussed in Section 2.1. In Section 2.2, properties of $\mathcal{G}$ are investigated. The equivalence relationship introduced by $\mathcal{G}$ and a systematic partition of $\mathbb{R}^d$ is studied in Section 2.3.

Chapter 3 is devoted to the construction of two functions. In Section 3.1, we define a positive function g which introduces a time change later. Another function h is defined in Section 3.2 and its superharmonicity is established in Section 3.3. Other properties of this function are also demonstrated in these two sections.

Chapter 4 contains the main result of this thesis. We give the path-by-path construction of a diffusion X_t in Section 4.1 and then identify the infinitesimal generator of its semigroup in Section 4.2. The invariance and symmetry properties of this process are studied in Section 4.3 and 4.4. Finally, in Section 4.5, a Markov process W_t is constructed path-by-path through X_t. The infinitesimal generator of the semigroup of W_t is computed. The result demonstrates that W_t is an h-Brownian motion in the interior of its state space and is normally reflected on the boundary.

Several restrictions are imposed in the process of this construction. In Section 4.6, the possibility and difficulty of relaxing them are discussed.

CHAPTER 2
THE STRUCTURE OF THE STATE SPACE

2.1 Definitions and Basic Properties

In this section, we shall define a family of inversions on $\overline{\mathbb{R}^d}$, the one point compactification of d-dimensional real space. Each of these inversions is a homeomorphism of the space onto itself.

For any open ball $B \subset \mathbb{R}^d$, let ∂B be the boundary of B. From now on, unless otherwise specified, we shall refer to ∂B as the *sphere* of B.

Let us denote by $\overline{\mathbb{R}^d}$ the one-point compactification of $\mathbb{R}^d$:

$$\overline{\mathbb{R}^d} = \mathbb{R}^d \cup \{\infty\}$$

For any point $c \in \mathbb{R}^d$, let $B = B(c, 1)$ be an open ball of radius 1, centered at c. We define a map I: $\overline{\mathbb{R}^d} \to \overline{\mathbb{R}^d}$ as follows.

DEFINITION 2.1.1. $I : \overline{\mathbb{R}^d} \to \overline{\mathbb{R}^d}$ is defined as

$$\begin{aligned} I(x) &= c + \frac{x-c}{\|x-c\|^2} \qquad \text{for } x \notin \{c, \infty\} \\ I(c) &= \infty \\ I(\infty) &= c \end{aligned}$$

This map leaves each point on the sphere of B fixed while it maps each interior point of B into $\overline{B}^c$, and vice versa; i.e., if $x \in \overline{B}^c$, then $I(x) \in B$. Thus, I is an inversion with respect to the sphere of B.

From this definition, we can derive the following basic properties of I.

PROPOSITION 2.1.1. 1. I is idempotent, i.e.,

$$I^{-1} = I \tag{2.1}$$

2. For all $x \notin \{c, \infty\}$,

$$\|x - c\| \times \|I(x) - c\| = 1 \tag{2.2}$$

PROOF. These are direct consequences of the definition.

1. By definition, for all $x \notin \{c, \infty\}$

$$\begin{aligned} I(I(x)) &= I(c + \frac{x-c}{\|x-c\|^2}) \\ &= c + (c + \frac{x-c}{\|x-c\|^2} - c)/(\|c + \frac{x-c}{\|x-c\|^2} - c\|^2) \\ &= c + (\frac{x-c}{\|x-c\|^2})/(\frac{\|x-c\|^2}{\|x-c\|^4}) \\ &= x \end{aligned}$$

   Moreover, we have

$$I(I(c)) = I(\infty) = c$$

   and

$$I(I(\infty)) = I(c) = \infty$$

   So, (2.1) is true.

2. For any $x \notin \{c, \infty\}$, $0 < \|x - c\| < \infty$, $0 < \|I(x) - c\| < \infty$, and

$$\begin{aligned} \|x - c\| \times \|I(x) - c\| &= \|x - c\| \times \|c + \frac{x-c}{\|x-c\|^2} - c\| \\ &= \|x - c\| \times \frac{\|x-c\|}{\|x-c\|^2} \\ &= 1 \end{aligned}$$

   So, (2.2) is true.

□

In order to study diffusion processes later using the inversion I, we need to learn more about its properties. To do this, we must first prove the following two lemmas in geometry. Properties of I are studied in Propositions 2.1.2 and 2.1.3 and in the next section.

LEMMA 2.1.1. Suppose two straight lines passing through points A, D and B, C respectively, intersect at point O. See Figure 1.

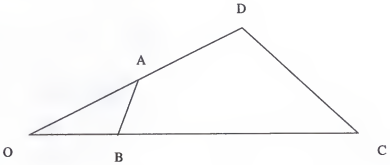

Figure 1

If

$$OB \cdot OC = OA \cdot OD$$

then

$$\frac{AB}{CD} = \frac{OA}{OC} = \frac{OB}{OD} \tag{2.3}$$

PROOF. Consider $\triangle OAB$ and $\triangle OCD$. By the given condition, we have

$$\frac{OA}{OC} = \frac{OB}{OD}$$

Since $\angle AOB = \angle COD$, we can conclude that $\triangle OAB$ and $\triangle OCD$ are similar. Hence

$$\frac{AB}{CD} = \frac{OA}{OC} = \frac{OB}{OD}$$

Obviously, this is also true in the special case when $OA = OD$.

□

LEMMA 2.1.2. Suppose two straight lines passing though points A, D and B, C respectively, intersect at point O (See Figure 2).

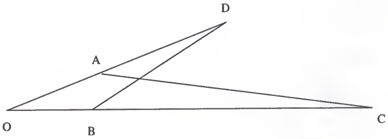

Figure 2

If

$$OB \cdot OC = OA \cdot OD$$

then

$$\frac{AC}{BD} = \frac{OA}{OB} = \frac{OC}{OD} \tag{2.4}$$

PROOF. Consider $\triangle OAC$ and $\triangle OBD$. By the given condition, we have

$$\frac{OA}{OB} = \frac{OC}{OD}$$

Since $\angle AOC = \angle BOD$, we can conclude that $\triangle OAC$ and $\triangle OBD$ are similar. Hence

$$\frac{AC}{BD} = \frac{OA}{OB} = \frac{OC}{OD}$$

Obviously, this is also true in the special case when $OA = OD$.

□

Equipped with Lemmas 2.1.1 and 2.1.2, we are ready to study the following properties of the inversion I.

PROPOSITION 2.1.2. For all x, $y \notin \{c, \infty\}$,

$$||I(x) - I(y)|| = \frac{||x - y||}{||x - c|| \times ||y - c||} \tag{2.5}$$

Proof. Depending on the relative positions of x and y, the proof can be divided into the following cases.

1. If both x and y are on the sphere of B, we have $I(x) = x$, $I(y) = y$ and $||x - c|| = ||y - c|| = 1$. The proof is trivial.

2. If both $x, y \in \overline{B}$ and at least one of them is not on the sphere, we have two triangles formed by points $\{c, x, y\}$ and $\{c, I(x), I(y)\}$, respectively, as shown in Figure 3.

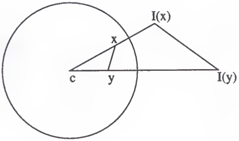

Figure 3

By (2.2) $||I(x) - c|| \times ||x - c|| = 1 = ||I(y) - c|| \times ||y - c||$. It follows from Lemma 2.1.1 that these two triangles are similar; hence

$$\frac{||x - y||}{||I(x) - I(y)||} = \frac{||y - c||}{||I(x) - c||}$$

or, equivalently,

$$\frac{||I(x) - I(y)||}{||x - y||} = \frac{||I(x) - c||}{||y - c||}$$

Since

$$||I(x) - c|| = \frac{1}{||x - c||}$$

we get

$$\frac{||I(x) - I(y)||}{||x - y||} = \frac{1}{||x - c|| \times ||y - c||}$$

Therefore,

$$||I(x) - I(y)|| = \frac{||x - y||}{||x - c|| \times ||y - c||}$$

3. If both $x, y \in B^c$, and at least on of them is not on the sphere, then the two triangles formed by $\{c, x, y\}$ and $\{c, I(x), I(y)\}$, respectively, are as shown in Figure 4.

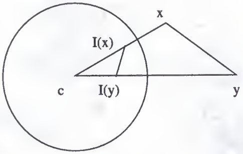

Figure 4

In this case, we also have $\|I(x) - c\| \times \|x - c\| = 1 = \|I(y) - c\| \times \|y - c\|$.

Lemma 2.1.1 can be applied here and we have

$$\frac{\|I(x) - I(y)\|}{\|x - y\|} = \frac{\|I(x) - c\|}{\|y - c\|}$$

By similar argument, we get

$$\|I(x) - I(y)\| = \frac{\|x - y\|}{\|x - c\| \times \|y - c\|}$$

4. If only one of the points is in B, then the triangles formed by $\{c, x, y\}$ and $\{c, I(x), I(y)\}$, respectively, are as shown in Figure 5.

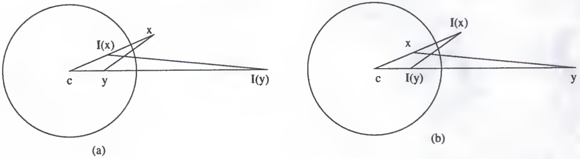

Figure 5

In either case, $\|I(x) - c\| \times \|x - c\| = 1 = \|I(y) - c\| \times \|y - c\|$, by (2.2). According to Lemma 2.1.2, we have

$$\frac{\|I(x) - I(y)\|}{\|x - y\|} = \frac{\|I(x) - c\|}{\|y - c\|}$$

Hence, by the same argument as above, we have

$$\|I(x) - I(y)\| = \frac{\|x - y\|}{\|x - c\| \times \|y - c\|}$$

□

PROPOSITION 2.1.3. For any $x,\ y \notin \{c,\ \infty\}$,

$$\frac{\|I(x) - y\|}{\|x - I(y)\|} = \frac{\|y - c\|}{\|x - c\|} \tag{2.6}$$

PROOF. There are four possible cases defined by the relative positions of x and y. They are:

(a) $x \in B$ and $y \notin B$;

(b) $x \notin B$ and $y \in B$;

(c) $x \in B$ and $y \in B$;

(d) $x \notin B$ and $y \notin B$;

as shown in Figure 6.

In each case, we have

$$\|I(y) - c\| \times \|y - c\| = 1 = \|I(x) - c\| \times \|x - c\|$$

Applying Lemma 2.1.1 in case (a) and (b), or Lemma 2.1.2 in cases (c) and (d), we have

$$\frac{\|I(x) - y\|}{\|x - I(y)\|} = \frac{\|y - c\|}{\|x - c\|}$$

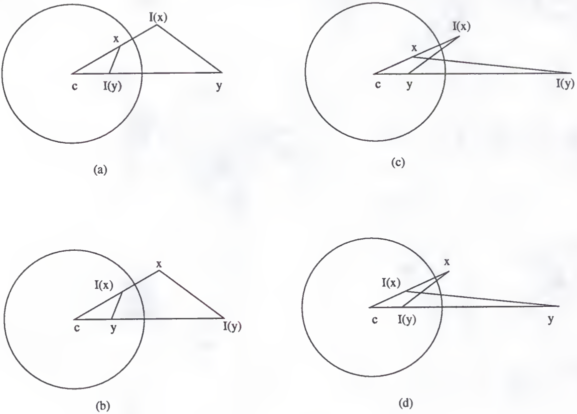

Figure 6

Therefore, we have the identity (2.6).

□

PROPOSITION 2.1.4. I is a continuous map from $\overline{\mathbb{R}^d}$ to $\overline{\mathbb{R}^d}$.

PROOF. Let us verify directly that I is continuous at c and ∞.

First, suppose a sequence $\{x_n\}_{n=1}^{\infty}$ in $\mathbb{R}^d$ has point ∞ as its limit point. Then $\|x_n\| \to \infty$ as $n \to \infty$. By (2.2), for any n,

$$\begin{aligned} \|I(x_n) - I(\infty)\| &= \|I(x_n) - c\| \\ &= \frac{1}{\|x_n - c\|} \\ &\leq \frac{1}{\|x_n\| - \|c\|} \end{aligned}$$

Since $\|x_n\| \to \infty$ as $n \to \infty$, the right hand side approaches zero. Hence, I is continuous at ∞.

Next, suppose a sequence $\{x_n\}_{n=1}^{\infty}$ in $\mathbb{R}^d$ converges to point c, i.e., $||x_n - c|| \to 0$ as $n \to \infty$. Then, $||I(x_n)|| + ||c|| \geq ||I(x_n) - c|| = ||x_n - c||^{-1} \to \infty$. Since $||c||$ is finite, we have $||I(x_n)|| \to \infty$, i.e., $I(x_n) \to I(c)$. Therefore, I is also continuous at point c.

Generally, let $\{x_n\}_{n=1}^{\infty}$ be a sequence in $\mathbb{R}^d$ that converges to $x_0 \neq c$. Then $||x_0 - c|| > 0$ and there exists $\epsilon > 0$ such that $||x_n - c|| > \epsilon$ for all n. Since $||x_n - x_0|| \to 0$ as $n \to \infty$, by (2.2),

$$||I(x_n) - I(x_0)|| = \frac{||x_n - x_0||}{||x_n - c|| \times ||x_0 - c||} \leq \frac{||x_n - x_0||}{\epsilon ||x_0 - c||} \to 0$$

as $n \to \infty$. So, I is continuous.

☐

The definition of I and Proposition 2.1.1 guarantee that I is a bijection. Thus, I is a homeomorphism of $\overline{\mathbb{R}^d}$ onto itself.

Generally, for each $B_j = B(c_j, 1),\ j = 1, 2, ..., \text{n}$, we can define an inversion I_j with respect to ∂B_j, and all of the properties discussed above hold for each inversion, respectively.

2.2 The Group $\mathcal{G}$ Generated By Inversions

In this section, a finite number of inversions are defined in the same way as in section 2.1, and the group of homeomorphisms of $\overline{\mathbb{R}^d}$ generated by these inversions is discussed.

Let $B_j = B(c_j, 1),\ j = 1, 2, \ldots, n$, be separated open balls in $\mathbb{R}^d$. By this, we mean $\overline{B_i} \cap \overline{B_j} = \emptyset$, for $i \neq j$. Let I_j be the inversion with respect to ∂B_j, the sphere of B_j, defined in the same way as I in section 2.1, for $j = 1, 2, \ldots, n$.

Each inversion I_j is a homeomorphism of $\overline{\mathbb{R}^d}$ onto itself, so they generate a non-Abelian group $\mathcal{G}$ of homeomorphisms on $\overline{\mathbb{R}^d}$.

Next, we shall prove a fact concerning the identity of this group.

LEMMA 2.2.1. Let

$$S = I_{n_1} I_{n_2} \dots I_{n_{k-1}} I_{n_k}$$

where $k \geq 1$ and $n_j \in \{1, 2, \dots, n\}$ for $1 \leq j \leq k$. If $n_i \neq n_{i+1}$ for $1 \leq j \leq k-1$, then $S \neq e$.

PROOF. By the definition of I_j, for any $x \notin \overline{B_j}$, we have $I_j(x) \in B_j$.

Suppose $S = I_{n_1} I_{n_2} \dots I_{n_{k-1}} I_{n_k}$ and $n_i \neq n_{i+1}$ for $1 \leq j \leq k-1$. Let $z = \infty$, $x_k = I_k(z)$ and

$$x_j = I_{n_j}(x_{j+1})$$

for $j = k-1, k-2, \dots, 3, 2, 1$. It follows that $x_k = I_{n_k}(\infty) \in B_{n_k}$. Since $n_k \neq n_{k-1}$, $x_k \notin \overline{B_{n_{k-1}}}$. Therefore, $x_{k-1} = I_{n_{k-1}}(x_k) \in B_{n_{k-1}}$. Repeating the same argument, we have

$$x_j = I_{n_j}(x_{j+1}) \in B_j$$

for all $1 \leq j \leq k-1$. Thus, $x_1 = S(z) \in B_{n_1}$, while $\infty \notin B_{n_1}$, and we conclude that $S \neq e$.

□

Since $\mathcal{G}$ is generated by $\{I_1, I_2, \dots, I_n\}$, any $S \in \mathcal{G} - \{e\}$, can be represented as a finite product of these inversions.

By the idempotent property of the inversions, for any $S \in \mathcal{G}$, if one representation of S is

$$S = I_{n_1} I_{n_2} \dots I_{n_{k-1}} I_{n_k}$$

we can always find an equivalent representation

$$S = I_{n'_1} I_{n'_2} \dots I_{n'_{k'-1}} I_{n'_{k'}}$$

such that $n'_i \neq n'_{i+1}$ for every $1 \leq i \leq k' - 1$.

Next we shall prove that this representation is unique.

LEMMA 2.2.2. If $S \in \mathcal{G} - \{e\}$ has a representation $S = I_{n_1}I_{n_2}\dots I_{n_{k-1}}I_{n_k}$ for some $n_1,\ n_2,\ \dots,\ n_k$ and $n_i \neq n_{i+1}$ for every $1 \leq i \leq k-1$, then this representation is unique.

PROOF. This lemma can be proved by way of contradiction.

Suppose this is not true, and we have another representation

$$S = I_{m_1}I_{m_2}\dots I_{m_{p-1}}I_{m_p} = I_{n_1}I_{n_2}\dots I_{n_{k-1}}I_{n_k}$$

where $m_i \neq m_{i+1}$ for every $1 \leq i \leq p-1$ and $n_i \neq n_{i+1}$ for every $1 \leq i \leq k-1$. Without loss of generality, we may also assume that $p \leq k$.

We claim $I_{n_1} = I_{m_1}$, or else we shall have a contradiction.

Since $S = I_{m_1}I_{m_2}\dots I_{m_{p-1}}I_{m_p}$, and every inversion is idempotent, it follows that

$$S^{-1} = I_{m_p}I_{m_{p-1}}\dots I_{m_2}I_{m_1}$$

If $I_{n_1} \neq I_{m_1}$, then the following product

$$T = I_{m_p}I_{m_{p-1}}\dots I_{m_2}I_{m_1}I_{n_1}I_{n_2}\dots I_{n_{k-1}}I_{n_k}$$

has the property that no two consecutive inversions are the same. Therefore, by Lemma 2.2.1, $T \neq e$. On the other hand, $T = S^{-1}S = e$. This contradiction shows that $I_{n_1} = I_{m_1}$.

Now we have

$$I_{m_2}\dots I_{m_{p-1}}I_{m_p} = I_{n_2}\dots I_{n_{k-1}}I_{n_k}$$

By mathematical induction,

$$I_{n_i} = I_{m_i}$$

for all $1 \leq i \leq p$.

If $p = k$, the proof is complete. If $p < k$, we have

$$e = I_{n_{p+1}}\dots I_{n_k}$$

But since $n_i \neq n_{i+1}$ for every $1 \leq i \leq k-1$, this is impossible according to Lemma 2.2.1.

Hence, the representation of any $S \in \mathcal{G} - \{e\}$ is unique.

□

From now on, unless otherwise specified, we always refer to this unique representation whenever S is mentioned. Now that we have the unique representation of any element of $\mathcal{G} - \{e\}$, we can define its length.

Definition 2.2.1. For any $S = I_{n_1}I_{n_2}\dots I_{n_{k-1}}I_{n_k} \in \mathcal{G} - \{e\}$, the *length* L of S is defined as the number of inversions in its unique representation. $L(S) = k$ for $S = I_{n_1}I_{n_2}\dots I_{n_{k-1}}I_{n_k}$. If we define $L(e) = 0$, then $L : \mathcal{G} \to \mathbb{Z}$.

The following is a generalization of Proposition 2.1.3.

Proposition 2.2.1. Let $S \in \mathcal{G} - \{e\}$. For any $x,\ y \notin \{c_1, c_2, \dots, c_n, \infty\}$,

$$\frac{\|S(x) - y\|}{\|x - S^{-1}(y)\|} = \frac{\|y - S(\infty)\|}{\|x - S^{-1}(\infty)\|} \tag{2.7}$$

Proof. Suppose $S = I_{n_1}I_{n_2}\dots I_{n_{k-1}}I_{n_k}$ is its unique representation. The conclusion can be proved by mathematical induction on the length of S.

(a) In the case of $S = I_{n_1}$, it follows that $S^{-1} = I_{n_1}$ and $S(\infty) = S^{-1}(\infty) = c_{n_1}$, and Proposition 2.1.3 can be applied directly to get the desired identity.

(b) Assume the identity is true for S_1 of length $k-1$; i.e., for $S_1 = I_{n_1}I_{n_2}\dots I_{n_{k-1}}$, we have

$$\frac{\|S_1(x) - y\|}{\|x - S_1^{-1}(y)\|} = \frac{\|y - S_1(\infty)\|}{\|x - S_1^{-1}(\infty)\|}$$

Let $S = S_1 I_{n_k}$, where $n_k \neq n_{k-1}$. Then, $S^{-1} = I_{n_k}S_1^{-1}$, and, by the inductive assumption,

$$\begin{aligned} \|S(x) - y\| &= \|S_1 I_{n_k}(x) - y\| \\ &= \|I_{n_k}(x) - S_1^{-1}(y)\| \times \frac{\|y - S_1(\infty)\|}{\|I_{n_k}(x) - S_1^{-1}(\infty)\|} \\ &= \|y - S_1(\infty)\| \times \frac{\|I_{n_k}(x) - S_1^{-1}(y)\|}{\|I_{n_k}(x) - S_1^{-1}(\infty)\|} \end{aligned}$$

It follows from (2.6) that

$$\|I_{n_k}(x) - S_1^{-1}(y)\| = \|x - I_{n_k}S_1^{-1}(y)\| \times \frac{\|S_1^{-1}(y) - I_{n_k}(\infty)\|}{\|x - I_{n_k}(\infty)\|}$$

and

$$\|I_{n_k}(x) - S_1^{-1}(\infty)\| = \|x - I_{n_k}S_1^{-1}(\infty)\| \times \frac{\|S_1^{-1}(\infty) - I_{n_k}(\infty)\|}{\|x - I_{n_k}(\infty)\|}$$

Hence, the ratio

$$\frac{\|I_{n_k}(x) - S_1^{-1}(y)\|}{\|I_{n_k}(x) - S_1^{-1}(\infty)\|} = \frac{\|x - S^{-1}(y)\| \times \|S_1^{-1}(y) - I_{n_k}(\infty)\|}{\|x - S^{-1}(\infty)\| \times \|S_1^{-1}(\infty) - I_{n_k}(\infty)\|}$$

It then follows that

$$\begin{aligned}\|S(x) - y\| &= \|y - S_1(\infty)\| \times \frac{\|x - S^{-1}(y)\|}{\|x - S^{-1}(\infty)\|} \times \frac{\|S_1^{-1}(y) - I_{n_k}(\infty)\|}{\|S_1^{-1}(\infty) - I_{n_k}(\infty)\|} \\ &= \frac{\|x - S^{-1}(y)\|}{\|x - S{-}1(\infty)\|} \times \frac{\|S_1^{-1}(y) - I_{n_k}(\infty)\| \times \|y - S_1(\infty)\|}{\|S_1^{-1}(\infty) - I_{n_k}(\infty)\|}\end{aligned}$$

Applying the assumption again for S_1^{-1}, we have

$$\begin{aligned}\|S_1^{-1}(y) - I_{n_k}(\infty)\| &= \|y - S_1 I_{n_k}(\infty)\| \times \frac{\|I_{n_k}(\infty) - S_1^{-1}(\infty)\|}{\|y - S_1(\infty)\|} \\ &= \|y - S(\infty)\| \times \frac{\|S_1^{-1}(\infty) - I_{n_k}(\infty)\|}{\|y - S_1(\infty)\|}\end{aligned}$$

So,

$$\|S_1^{-1}(y) - I_{n_k}(\infty)\| \times \|y - S_1(\infty)\| = \|y - S(\infty)\| \times \|S_1^{-1}(\infty) - I_{n_k}(\infty)\|$$

Thus,

$$\begin{aligned}\|S(x) - y\| &= \frac{\|x - S^{-1}(y)\|}{\|x - S^{-1}(\infty)\|} \times \frac{\|y - S(\infty)\| \times \|S_1^{-1}(\infty) - I_{n_k}(\infty)\|}{\|S_1^{-1}(\infty) - I_{n_k}(\infty)\|} \\ &= \|x - S^{-1}(y)\| \times \frac{\|y - S(\infty)\|}{\|x - S^{-1}(\infty)\|}\end{aligned}$$

Therefore, we have (2.7).

So, by mathematical induction, identity (2.7) is true for S of any finite length.

$\square$

Similarly, Proposition 2.1.2 can also be generalized for general $S \in \mathcal{G} - \{e\}$. In doing so, for any $S \in \mathcal{G} - \{e\}$, we introduce a factor D_S. This factor is useful in our understanding of the properties of the group $\mathcal{G}$; it is also important in the construction of a superharmonic function h that will be discussed in the following chapter.

PROPOSITION 2.2.2. Let $S = I_{n_1} I_{n_2} \dots I_{n_{k-1}} I_{n_k}$. For any $x, y \notin \{c_{n_1},\ c_{n_2},\ \dots,\ c_{n_k},\ \infty\}$, the following identity is true.

$$\|S(x) - S(y)\| = \frac{\|x-y\|}{\|x - S^{-1}(\infty)\| \times \|y - S^{-1}(\infty)\| \times {D_S}^2} \tag{2.8}$$

where, for $k = 1$,

$$D_S = 1; \tag{2.9}$$

and for $k \geq 2$,

$$\begin{aligned} D_S \;=\; & \|I_{n_1}(\infty) - I_{n_2}(\infty) \dots I_{n_k}(\infty)\| \times \|I_{n_2}(\infty) - I_{n_3}(\infty) \dots I_{n_k}(\infty)\| \times \cdots \times \\ & \|I_{n_{k-2}}(\infty) - I_{n_{k-1}}(\infty) I_{n_k}(\infty)\| \times \|I_{n_{k-1}}(\infty) - I_{n_k}(\infty)\| \end{aligned} \tag{2.10}$$

PROOF. This result can be proved by mathematical induction on the length of S.

(a) In the case of $k = 1$, $S = I_j$, $j \in \{1, 2, \dots, n\}$, we have Proposition 2.1.2, i.e.,

$$\|I_j(x) - I_j(y)\| = \frac{\|x-y\|}{\|x - I_j(\infty)\| \times \|y - I_j(\infty)\|}$$

(b) In the case of $k = 2$, suppose $S = I_i I_j$, where $i, j \in \{1, 2, \dots, n\}$. It follows that $S^{-1} = I_j I_i$ and $D_S = \|I_i(\infty) - I_j(\infty)\|$.

By Proposition 2.1.2, we have

$$\begin{aligned} \|S(x) - S(y)\| &= \|I_i I_j(x) - I_i I_j(y)\| \\ &= \frac{\|I_j(x) - I_j(y)\|}{\|I_j(x) - I_i(\infty)\| \times \|I_j(y) - I_i(\infty)\|} \end{aligned}$$

Using Proposition 2.1.2 again, we observe

$$\|I_j(x) - I_j(y)\| = \frac{\|x-y\|}{\|x - I_j(\infty)\| \times \|y - I_j(\infty)\|}$$

Moreover, by Proposition 2.1.3

$$\begin{aligned} \|I_j(x) - I_i(\infty)\| &= \|x - I_jI_i(\infty)\| \times \frac{\|I_i(\infty) - I_j(\infty)\|}{\|x - I_j(\infty)\|} \\ \|I_j(y) - I_i(\infty)\| &= \|y - I_jI_i(\infty)\| \times \frac{\|I_i(\infty) - I_j(\infty)\|}{\|y - I_j(\infty)\|} \end{aligned}$$

Combining these two expressions, we have

$$\begin{aligned} & \|I_j(x) - I_i(\infty)\| \times \|I_j(y) - I_i(\infty)\| \\ = \; & \|x - I_jI_i(\infty)\| \times \|y - I_jI_i(\infty)\| \times \frac{\|I_i(\infty) - I_j(\infty)\|^2}{\|x - I_j(\infty)\| \times \|y - I_j(\infty)\|} \end{aligned}$$

Therefore,

$$\begin{aligned} \|S(x) - S(y)\| &= \frac{\|x-y\|}{\|x - I_j(\infty)\| \times \|y - I_j(\infty)\|} \times \\ & \quad \frac{\|x - I_j(\infty)\| \times \|y - I_j(\infty)\|}{\|x - I_jI_i(\infty)\| \times \|y - I_jI_i(\infty)\| \times \|I_i(\infty) - I_j(\infty)\|^2} \\ &= \frac{\|x-y\|}{\|x - I_jI_i(\infty)\| \times \|y - I_jI_i(\infty)\| \times \|I_i(\infty) - I_j(\infty)\|^2} \\ &= \frac{\|x-y\|}{\|x - S^{-1}(\infty)\| \times \|y - S^{-1}(\infty)\| \times {D_S}^2} \end{aligned}$$

(c) Assuming that identity (2.8) is true for S of length $k-1$, i.e., for $S_1 = I_{n_2} \dots I_{n_k}$, we have

$$\|S_1(x) - S_1(y)\| = \frac{\|x-y\|}{\|x - S_1^{-1}(\infty)\| \times \|y - S_1^{-1}(\infty)\| \times D_{S_1}^2}$$

Let $S = I_{n_1}S_1 = I_{n_1}I_{n_2} \dots I_{n_k}$, where $n_1 \neq n_2$. Then we have

$$S^{-1} = S_1^{-1} I_{n_1}$$

and, by (2.10)

$$\begin{aligned} D_S &= \|I_{n_1}(\infty) - I_{n_2}(\infty) \dots I_{n_k}(\infty)\| \times \|I_{n_2}(\infty) - I_{n_3}(\infty) \dots I_{n_k}(\infty)\| \times \cdots \times \\ & \quad \|I_{n_{k-2}}(\infty) - I_{n_{k-1}}(\infty) I_{n_k}(\infty)\| \times \|I_{n_{k-1}}(\infty) - I_{n_k}(\infty)\| \\ &= \|I_{n_1} - S_1(\infty)\| \times D_{S_1} \end{aligned}$$

Now, by Proposition 2.1.2

$$\begin{aligned} \|S(x) - S(y)\| &= \|I_{n_1}S_1(x) - I_{n_1}S_1(y)\| \\ &= \frac{\|S_1(x) - S_1(y)\|}{\|S_1(x) - I_{n_1}(\infty)\| \times \|S_1(y) - I_{n_1}(\infty)\|} \end{aligned}$$

and by the inductive assumption,

$$\|S_1(x) - S_1(y)\| = \frac{\|x-y\|}{\|x - S_1^{-1}(\infty)\| \times \|y - S_1^{-1}(\infty)\| \times {D_{S_1}}^2}$$

By Proposition 2.2.1

$$\begin{aligned} \|S_1(x) - I_{n_1}(\infty)\| &= \|x - S_1^{-1}I_{n_1}(\infty)\| \times \frac{\|I_{n_1}(\infty) - S_1(\infty)\|}{\|x - S_1^{-1}(\infty)\|} \\ \|S_1(y) - I_{n_1}(\infty)\| &= \|y - S_1^{-1}I_{n_1}(\infty)\| \times \frac{\|I_{n_1}(\infty) - S_1(\infty)\|}{\|y - S_1^{-1}(\infty)\|} \end{aligned}$$

Substituting the above expressions, we obtain

$$\begin{aligned} \|S(x) - S(y)\| &= \frac{\|x-y\|}{\|x - S_1^{-1}(\infty)\| \times \|y - S_1^{-1}(\infty)\| \times {D_{S_1}}^2} \times \\ &\quad \frac{\|x - S_1^{-1}(\infty)\|}{\|x - S_1^{-1}I_{n_1}(\infty)\| \times \|I_{n_1}(\infty) - S_1(\infty)\|} \times \\ &\quad \frac{\|y - S_1^{-1}(\infty)\|}{\|y - S_1^{-1}I_{n_1}(\infty)\| \times \|I_{n_1}(\infty) - S_1(\infty)\|} \\ &= \frac{\|x-y\|}{\|x - S^{-1}(\infty)\| \times \|y - S^{-1}(\infty)\| \times {D_{S_1}}^2 \times \|I_{n_1}(\infty) - S_1(\infty)\|^2} \\ &= \frac{\|x-y\|}{\|x - S^{-1}(\infty)\| \times \|y - S^{-1}(\infty)\| \times {D_S}^2} \end{aligned}$$

Therefore, identity (2.8) is true for all $S \in \mathcal{G}$.

□

2.3 Orbits and the Horizon Set

As we have shown in section 2.1, any single map I_j defines an inversion in $\overline{\mathbb{R}^d}$. We are now ready to investigate the geometry of these inversions, particularly, the geometric characteristics of elements of the group $\mathcal{G}$. The following construction is based on the same method first given in [5] by Glover and Rao in the 2-dimensional case.

The group $\mathcal{G}$ of homeomorphisms on $\overline{\mathbb{R}^d}$ defines an equivalence relationship in $\overline{\mathbb{R}^d}$ in the following way:

$$x \sim y \text{ if and only if } y = S(x) \text{ for some } S \in \mathcal{G}$$

An equivalence class is formally defined as an *orbit* below.

DEFINITION 2.3.1. For any $x \in \overline{\mathbb{R}^d}$, the *orbit* of x, denoted by $[x]$, is defined as the following:

$$[x] = \{S(x) : \ S \in \mathcal{G}\}$$

Since each element of the group $\mathcal{G}$ is a homeomorphism on $\overline{\mathbb{R}^d}$, every point in $\overline{\mathbb{R}^d}$ belongs to a certain orbit. Proposition 2.3.1 below characterizes the collection of all orbits. First, we need a lemma.

LEMMA 2.3.1. For every $x \in \overline{\mathbb{R}^d} - \cup_{j=1}^n B_j$, and for every $S \in \mathcal{G}$ such that $L(S) \geq 2$, we have

$$S(x) \in \cup_{j=1}^n B_j$$

PROOF. For any $j \in \{1, 2, \ldots, n\}$, let I_j be the inversion defined in Definition 2.1.1 with respect to ∂B_j, where $B_j = B(c_j, 1)$.

By definition, if $x \in \partial B_j$, then $I_j(x) = x$; otherwise, for any $x \notin \overline{B_j}$, $I_j(x) \in B_j$.

First, suppose $S = I_{n_{k-1}} I_{n_k}$. For every $x \in \overline{\mathbb{R}^d} - \cup_{j=1}^n B_j$, there are two possibilities. If $x \notin \partial B_{n_k}$, then $I_{n_k}(x) \in B_{n_k}$. Otherwise, if $x \in \partial B_{n_k}$, then $I_{n_k}(x) = x \in \partial B_{n_k}$.

But since $n_{k-1} \neq n_k$, by our assumption,

$$\overline{B_{n_{k-1}}} \cap \overline{B_{n_k}} = \emptyset$$

Therefore, in either case, $I_{n_k}(x) \notin \overline{B_{n_{k-1}}}$. So, $I_{n_{k-1}}I_{n_k}(x) \in B_{n_{k-1}}$.

Now we can prove the general case by mathematical induction. Assume for any $S_k = I_{n_1}I_{n_2}\dots I_{n_{k-1}}I_{n_k}$ with $2 \leq k \leq N$, it is true that $S_k(x) \in B_{n_1}$, for any $x \in \overline{\mathbb{R}^d} - \cup_{j=1}^{n} B_j$. Then for $S = I_{n_0}I_{n_1}I_{n_2}\dots I_{n_{N-1}}I_{n_N}$, where $n_0 \neq n_1$, let

$$y = I_{n_1}I_{n_2}\dots I_{n_{N-1}}I_{n_N}(x)$$

we have, by our assumption, $y \in B_{n_1}$. Hence, $y \notin \overline{B_{n_0}}$. It follows that

$$S(x) = I_{n_0}(y) \in B_{n_0} \subset \cup_{j=1}^{n} B_j$$

By mathematical induction, for every $S \in \mathcal{G}$, and every $x \in \overline{\mathbb{R}^d} - \cup_{j=1}^{n} B_j$, we have $S(x) \in \cup_{j=1}^{n} B_j$.

□

This lemma leads to the following conclusion.

PROPOSITION 2.3.1. Each orbit contains at most one point in $\overline{\mathbb{R}^d} - \cup_{j=1}^{n} B_j$.

PROOF. Suppose both x and y are in $\overline{\mathbb{R}^d} - \cup_{j=1}^{n} B_j$, and they are both in the same orbit, i.e., $[x] = [y]$. Then there exists $S \in \mathcal{G}$ with $S(x) = y$. We can show that $x = y$.

We discuss all possible cases according to the length of S.

First, by Lemma 2.3.1, the case of $L(S) \geq 2$ is impossible since

$$x \in \overline{\mathbb{R}^d} - \cup_{j=1}^{n} B_j$$

would imply

$$y = S(x) \in \cup_{j=1}^{n} B_j$$

Second, let us assume that $L(S) = 1$, so $S = I_k$, for some $k \in \{1, 2, \ldots, n\}$. If $x \notin \partial B_k$, then $y = I_k(x) \in B_k$, but this contradicts the assumption that

$$y \in \overline{\mathbb{R}^d} - \cup_{j=1}^{n} B_j$$

So we must have $x \in \partial B_k$. It then follows that

$$y = S(x) = I_k(x) = x$$

Finally, if $L(S) = 0$, then $S = e$, so $x = y$. □

In general, there are two types of orbits. The first type contains a point in $\overline{\mathbb{R}^d} - \cup_{j=1}^{n} B_j$ and the second type does not contain such a point. Letting

$$E = \overline{\mathbb{R}^d} - \cup_{j=1}^{n} B_j$$

we can define the horizon set.

DEFINITION 2.3.2. The horizon H is defined as

$$H = \{x \in \overline{\mathbb{R}^d} : [x] \cap E = \emptyset\}$$

Restricting each homeomorphism of $\mathcal{G}$ to H^c, we can see that the group $\mathcal{G}$ defines an equivalence relation on H^c and that each equivalence class contains exactly one point in E. Let $H^c/\mathcal{G}$ be the collection of all equivalence classes, or orbits, in H^c. For any $[x]$ in $H^c/\mathcal{G}$, let y be this unique point in E. If we define $\phi : H^c/\mathcal{G} \to E$ by

$$\phi([x]) = y$$

then ϕ is a bijection from $H^c/\mathcal{G}$ onto E. Moreover, if we define a map $\Phi : H^c \to E$ by

$$\Phi(x) = \phi([x]) = y$$

then Φ is a map from H^c onto E.

Next we proceed to partition $\overline{\mathbb{R}^d}$ into the union of a sequence of regions. Define

$$\begin{aligned} E_S &= \{S(x) : x \in E\} \text{ for any } S \in \mathcal{G} \\ \mathcal{G}_k &= \{S \in \mathcal{G} : L(S) = k\} \\ F_k &= \cup_{S \in \mathcal{G}_k} E_S \end{aligned}$$

Since E is a closed set and every $S \in \mathcal{G}$ is continuous, E_S is also closed.

For $k = 0$, $\mathcal{G}_k = \{e\}$; and for every $k \geq 1$, the cardinality of $\mathcal{G}_k$ is $n(n-1)^{k-1}$. Therefore, F_k is a closed set since it is a finite union of closed sets.

The following lemma further characterizes the horizon set and its complement.

LEMMA 2.3.2. $H = \cap_{S \in \mathcal{G}} S(\cup_{j=1}^n B_j)$ is compact and

$$H^c = \cup_{S \in \mathcal{G}} E_S$$

PROOF. It is better to prove the result for H^c first. By the definition of H,

$$H^c = \{x \in \overline{\mathbb{R}^d} : [x] \cap E \neq \emptyset\}$$

Therefore, for any $S \in \mathcal{G}$, and any $x \in E_S$, we have $S^{-1}(x) \in E$ by the definition of E_S. So, $[\, x \,] \cap E \neq \emptyset$. Thus, $x \in H^c$ for all $x \in E_S$. Therefore, $E_S \subset H^c$ for all $S \in \mathcal{G}$. This implies that

$$\cup_{S \in \mathcal{G}} E_S \subset H^c$$

On the other hand, for any $x \in H^c$, $[x] \cap E \neq \emptyset$. So, there exists $S \in \mathcal{G}$ such that $y = S(x) \in E$. Since $x = S^{-1}(y)$ and $y \in E$, we know that $x \in E_{S^{-1}}$. Therefore,

$$H^c \subset \cup_{S \in \mathcal{G}} E_S$$

Thus we have the desired result.

Next, let us examine the set $(E_S)^c$ for an arbitrary $S \in \mathcal{G}$. By definition, $E_S = S(E)$. Since S is a bijection of $\overline{\mathbb{R}^d}$ onto itself,

$$\overline{\mathbb{R}^d} = S(\overline{\mathbb{R}^d}) = S(E \cup E^c) = S(E) \cup S(E^c)$$

i.e., $(S(E))^c = S(E^c)$. So we have

$$(E_S)^c = (S(E))^c = S(E^c) = S(\cup_{j=1}^n B_j)$$

Hence,

$$H = (H^c)^c = \cap_{S\in\mathcal{G}} (E_S)^c = \cap_{S\in\mathcal{G}} S(\cup_{j=1}^n B_j)$$

Notice, if $S' \in \mathcal{G}$ and $L(S) < L(S')$, then

$$S'(\cup_{j=1}^n B_j) \subset S'(\cup_{j=1}^n \overline{B_j}) \subset S(\cup_{j=1}^n B_j).$$

So,

$$H = \cap_{S\in\mathcal{G}} S(\cup_{j=1}^n \overline{B_j})$$

Therefore, H is closed. Since it is also bounded, it is compact.

□

Since $\mathcal{G} = \cup_{k=0}^\infty \mathcal{G}_k$, it follows from Lemma 2.3.2 that

$$H^c = \cup_{S\in\mathcal{G}} E_S = \cup_{k=0}^\infty \cup_{S\in\mathcal{G}_k} E_S = \cup_{k=0}^\infty F_k$$

where $F_0 = E$.

For any $S \in \mathcal{G} - \{e\}$, if $S = I_{n_1} I_{n_2} \dots I_{n_{k-1}} I_{n_k}$, then let $P_1(S) = I_{n_1}$, the first component of its unique representation. For example, $P_1(I_3 I_1 I_2) = I_3$. Hence, for any $k \geq 1$,

$$F_k = \cup_{S\in\mathcal{G}_k} E_S = \cup_{j=1}^n \cup_{S\in\mathcal{G}_k, P_1(S)=I_j} E_S$$

where n is the total number of inversions defined in $\overline{\mathbb{R}^d}$.

Setting $F_k^j = \cup_{S\in\mathcal{G}_k, P_1(S)=I_j} E_S$ for any $k \geq 1$, we have

$$F_k = \cup_{j=1}^n F_k^j$$

and

$$H^c = \cup_{k=0}^\infty F_k = F_0 \cup (\cup_{k=1}^\infty \cup_{j=1}^n F_k^j).$$

This partition of H^c will be the basis for defining functions on H^c in the next chapter, and ultimately, the reflected diffusion.

The map Φ is also continuous, as shown in the next proposition. The proof of this proposition is similar to that of Proposition 2.11 in [5], except that since the boundaries of $B_j, j = 1, 2, ..., n$ are disjoint, the restriction on the domain of Φ in [5] is no longer necessary.

PROPOSITION 2.3.2. Φ is continuous on H^c.

PROOF. First of all, by the definition of the inversion, any member of $\mathcal{G}$ is a continuous map on $\mathbb{R}^d$, hence, on H^c.

For any point $x \in H^c$, either there exists some $S \in \mathcal{G}$ such that

$$S(x) \in \cup_{j=1}^{n} \partial B_j,$$

or there is no such S. We shall discuss these two cases separately.

First, if for some $1 \leq j \leq n$ and $S = I_{n_1} I_{n_2} \dots I_{n_k}$,

$$S(x) \in \partial B_j$$

then

$$x \in S^{-1}(\partial B_j)$$

and

$$\Phi(x) = \Phi \circ S(x) \in \partial B_j$$

since the boundaries of all balls are disjoint and $[x]$ has exactly one point in E. Because $S^{-1} \circ \Phi(x) = x$, there exists a neighborhood N of $\Phi(x)$ such that

$$N \subset E \cup (B_j - I_j(\cup_{i \neq j} B_i)).$$

Since Φ is the identity on E and $\Phi = I_j$ on $B_j - I_j(\cup_{i \neq j} B_i)$, we have

$$\Phi|_E(x) = x$$

and

$$\Phi|_{\overline{(B_j - I_j(\cup_{i \neq j} B_i))}}(x) = c_j + \frac{1}{\|x - c_j\|^2}(x - c_j)$$

For every $x \in \partial B_j$, $\|x - c_j\| = 1$. Then we have

$$\Phi|_{\overline{(B_j - I_j(\cup_{i \neq j} B_i))}}(x) = x$$

i.e., these two continuous functions coincide on ∂B_j. Therefore, on $S^{-1}(E)$, $\Phi = S$; on $S^{-1}(\overline{B_j - I_j(\cup_{i \neq j} B_i)})$, $\Phi = I_j S$. Both of these functions are continuous and they coincide on $S^{-1}(\partial B_j)$. Therefore, Φ is continuous on the open set $S^{-1}(N)$ containing x.

Second, if for any $S \in \mathcal{G}$,

$$S(x) \notin \cup_{j=1}^{n} \partial B_j,$$

then $\Phi(x) \notin \cup_{j=1}^{n} \partial B_j$. So $\Phi(x) \in E^o$, the interior of E. Since $[x]$ has exactly one point in E, there exists a unique $T \in \mathcal{G}$ such that $x = T \circ \Phi(x)$. So $\Phi(y) = T^{-1}(y)$ on $T(E^o)$ and it is continuous.

$\square$

CHAPTER 3
TIME CHANGE AND H-TRANSFORM FUNCTIONS

3.1 The Time Change Density g

Having completed the partition of $\mathbb{R}^d$, we are now ready to define a positive function g on H^c. This function g will then define a time change of the Brownian motion.

Recall that

$$\begin{aligned} F_k^j &= \cup_{S \in \mathcal{G}_k, P_1(S)=I_j} E_S, \quad \text{for } k \geq 1,\ 1 \leq j \leq n \\ F_0 &= E \\ F_k &= \cup_{j=1}^{n} F_k^j, \text{ for } k \geq 1 \\ H^c &= \cup_{k=0}^{\infty} F_k \end{aligned}$$

Notice that for any $j \in \{1, 2, \dots, n\}$,

$$\begin{aligned} I_j(F_1^j) &\subset F_0 \\ I_j(F_k^j) &\subset F_{k-1}, \text{ for } k = 1, 2, 3, \dots \end{aligned}$$

In the following definition, we shall define g on F_0 first. Then, inductively, we can extend the definition of g to all of H^c. Let us now state the definition of g first and then prove that it is a well-defined positive function. Define

$$\begin{aligned} &(a) \quad g(x) = 1, \quad x \in F_0; \\ &(b) \quad g(x) = \frac{1}{\|x - c_j\|^4} g(I_j(x)), \ x \in F_k^j;\ j \in \{1, 2, \dots, n\},\ k = 1, 2, 3, \dots \end{aligned}$$

PROPOSITION 3.1.1. The function g is well-defined and positive.

PROOF. By (a), g is defined on F_0. For all $x \in F_1 = \cup_{j=1}^{n} F_1^j$, if $x \in F_1^j$, $j \in \{1, 2, 3, \dots, n\}$, then

$$I_j(x) \in F_0$$

and by (b),

$$\begin{aligned} g(x) &= \frac{1}{\|x - c_j\|^4} g(I_j(x)) \\ &= \frac{1}{\|x - c_j\|^4} \end{aligned}$$

So g is well defined on $F_1 = \cup_{j=1}^{n} F_1^j$ and it is positive.

Suppose that g is defined on F_0, F_1, ..., F_{k-1} already. Then for any $x \in F_k^j$, $I_j(x)$ is in F_{k-1}, on which g is already defined. So $g(x)$ can be defined according to (b) for each $j = 1, 2, \dots, n$. Therefore, g is defined on $F_k = \cup_{j=1}^{n} F_k^j$. g is obviously positive since it is so in each previous inductive step.

□

The role of g will become clear in the next chapter, where it introduces a positive additive functional, which eventually leads to a time change of the diffusion process we study. The relationship between the function g and the inversions is demonstrated in Proposition 3.1.2 below.

PROPOSITION 3.1.2. $g(x)$ satisfies

$$g(x) = \frac{1}{\|x - c_j\|^4} g(I_j(x)) \tag{3.1}$$

for each $j = 1, 2, \dots, n$ and all $x \in H^c$.

PROOF. First, for every $x \in F_0 \subset H^c$, $I_j(x) \in F_1^j$. So, $g(x) = 1$ by (a) and

$$\begin{aligned} g(I_j(x)) &= \frac{1}{\|I_j(x) - c_j\|^4} g(I_j^2(x)) \\ &= \|x - c_j\|^4 g(x) \\ &= \|x - c_j\|^4 \end{aligned}$$

by (b). Therefore, (3.1) is true.

Second, suppose $x \in F_k$ for some $k \geq 1$, then $x \in F_k^i$ for some $1 \leq i \leq n$. There are two cases.

(i) If $i = j$, then $I_j(x) \in F_{k-1}$. According to (b), the left hand side is defined by the right hand side. So, the above identity is true.

(ii) If $i \neq j$, then $I_j(x) \in F_{k+1}^j$. By (b) ,

$$\begin{aligned} g(I_j(x)) &= \frac{1}{\|I_j(x) - c_j\|^4} g(I_j^2(x)) \\ &= \frac{1}{\|I_j(x) - c_j\|^4} g(x) \end{aligned}$$

So, the right hand side

$$\begin{aligned} \frac{1}{\|x - c_j\|^4} g(I_j(x)) &= \frac{1}{\|x - c_j\|^4} \times \frac{1}{\|I_j(x) - c_j\|^4} g(x) \\ &= \frac{1}{(\|x - c_j\| \times \|I_j(x) - c_j\|)^4} g(x) \\ &= g(x) \end{aligned}$$

Thus, the identity is proved.

□

3.2 The Function h and Its Properties

We now define a superharmonic function h which will be used to h-transform our Brownian motion.

For any $S \in \mathcal{G} - \{e\}$, let $\alpha_S = {D_S}^{-1}$, where D_S is defined as in Proposition 2.2.2. Also, let

$$C = \{S(\infty) : S \in \mathcal{G}\}.$$

Since $\mathcal{G}$ is countable, so is C. We define

$$h(x) = 1 + \sum_{S \in \mathcal{G} - \{e\}} \frac{\alpha_S^{d-2}}{\|x - S(\infty)\|^{d-2}}, \quad \text{for all } x \in \mathbb{R}^d - C.$$

Note that

$$\lim_{x\to z} h(x) = \infty$$

for every $z \in C$, so we may set $h(z) = \infty$ for every $z \in C$.

It is obvious from this definition that h is positive. In the next section, we shall prove it is finite in $\mathbb{R}^d$ except on the countable set C.

For the present, let us examine its property associated with inversions first.

LEMMA 3.2.1. For $S \neq I_j$, $D_S\|S(\infty) - I_j(\infty)\| = D_{I_jS}$, hence,

$$\alpha_{I_jS} = \frac{\alpha_S}{\|S(\infty) - I_j(\infty)\|}$$

PROOF. There are two possible cases, $P_1(S) = I_j$ and $P_1(S) \neq I_j$.

(i) If $P_1(S) = I_j$, suppose $S = I_jI_{n_2} \dots I_{n_k}$, $k \geq 2$. Then $I_jS = I_{n_2} \dots I_{n_k}$, so

$$\begin{aligned} & \|S(\infty) - I_j(\infty)\|D_S \\ = \; & \|I_jI_{n_2} \dots I_{n_k}(\infty) - I_j(\infty)\| \times \|I_j(\infty) - I_{n_2} \dots I_{n_k}(\infty)\| \times \\ & \|I_{n_2}(\infty) - I_{n_3} \dots I_{n_k}(\infty)\| \times \cdots \times \|I_{n_{k-1}}(\infty) - I_{n_k}(\infty)\| \\ = \; & \|I_{n_2}(\infty) - I_{n_3} \dots I_{n_k}(\infty)\| \times \cdots \times \|I_{n_{k-1}}(\infty) - I_{n_k}(\infty)\| \\ = \; & D_{I_jS} \end{aligned}$$

(ii) If $P_1(S) \neq I_j$, suppose $S = I_{n_2} \dots I_{n_k}$, $k \geq 2$. Then $I_jS = I_jI_{n_2} \dots I_{n_k}$, so

$$\begin{aligned} & \|I_j(\infty) - S(\infty)\|D_S \\ = \; & \|I_j(\infty) - I_{n_2} \dots I_{n_k}(\infty)\| \times \\ & \|I_{n_2}(\infty) - I_{n_3} \dots I_{n_k}(\infty)\| \times \cdots \times \|I_{n_{k-1}}(\infty) - I_{n_k}(\infty)\| \\ = \; & D_{I_jS} \end{aligned}$$

□

PROPOSITION 3.2.2. The function h satisfies

$$h(x) = \frac{1}{\|x - I_j(\infty)\|^{d-2}} h(I_j(x)), \quad \forall x \in \mathbb{R}^d - C, \;\; j = 1, 2, \ldots, n$$

PROOF. By the definition of h,

$$\begin{aligned} h(x) &= 1 + \frac{1}{\|x - I_j(\infty)\|^{d-2}} + \sum_{S \in \mathcal{G} - \{e, I_j\}} \frac{\alpha_S^{d-2}}{\|x - S(\infty)\|^{d-2}} \\ h(I_j(x)) &= 1 + \frac{1}{\|I_j(x) - I_j(\infty)\|^{d-2}} + \sum_{S \in \mathcal{G} - \{e, I_j\}} \frac{\alpha_S^{d-2}}{\|I_j(x) - S(\infty)\|^{d-2}} \end{aligned}$$

Recall that for any $S \in \mathcal{G} - \{e\}$, $P_1(S)$ is the first inversion in the unique representation of S. Using (2.2) for the second term in the above expression and Proposition 2.1.3 for the terms in the summation, we have

$$h(I_j(x)) = 1 + \|x - I_j(\infty)\|^{d-2} + \sum_{S \in \mathcal{G} - \{e, I_j\}} \frac{\alpha_S^{d-2} \|x - I_j(\infty)\|^{d-2}}{\|x - I_j S(\infty)\|^{d-2} \|S(\infty) - I_j(\infty)\|^{d-2}}$$

So,

$$\frac{1}{\|x - I_j(\infty)\|^{d-2}} h(I_j(x)) = 1 + \frac{1}{\|x - I_j(\infty)\|^{d-2}} + \sum_{S \in \mathcal{G} - \{e, I_j\}} \frac{\alpha_S^{d-2} / \|S(\infty) - I_j(\infty)\|^{d-2}}{\|x - I_j S(\infty)\|^{d-2}}$$

Let

$$\mathcal{G}^j = \{S : \; S \in \mathcal{G} - \{e, I_j\}, \; P_1(S) = I_j\}$$

and

$$\mathcal{G}^{j*} = \{S : \; S \in \mathcal{G} - \{e, I_j\}, \; P_1(S) \neq I_j\}$$

Then we have $\mathcal{G} - \{e, I_j\} = \mathcal{G}^j \cup \mathcal{G}^{j*}$. Moreover,

$$\mathcal{G}^j = \{I_j S : \; S \in \mathcal{G}^{j*}\}$$

and

$$\mathcal{G}^{j*} = \{I_j S : \; S \in \mathcal{G}^j\}$$

By Lemma 3.2.1,

$$\frac{\alpha_S}{\|S(\infty) - I_j(\infty)\|} = \alpha_{I_j S}$$

So, if we let $S' = I_jS$,

$$\begin{aligned}
&\sum_{S\in\mathcal{G}-\{e,I_j\}} \frac{\alpha_S^{d-2}/\|S(\infty)-I_j(\infty)\|^{d-2}}{\|x-I_jS(\infty)\|^{d-2}} \\
&= \sum_{S\in\mathcal{G}-\{e,I_j\}} \frac{\alpha_{I_jS}^{d-2}}{\|x-I_jS(\infty)\|^{d-2}} \\
&= \sum_{S\in\mathcal{G}^j} \frac{\alpha_{I_jS}^{d-2}}{\|x-I_jS(\infty)\|^{d-2}} + \sum_{S\in\mathcal{G}^{j*}} \frac{\alpha_{I_jS}^{d-2}}{\|x-I_jS(\infty)\|^{d-2}} \\
&= \sum_{S'\in\mathcal{G}^{j*}} \frac{\alpha_{S'}^{d-2}}{\|x-S'(\infty)\|^{d-2}} + \sum_{S'\in\mathcal{G}^{j}} \frac{\alpha_{S'}^{d-2}}{\|x-S'(\infty)\|^{d-2}} \\
&= \sum_{S'\in\mathcal{G}-\{e,I_j\}} \frac{\alpha_{S'}^{d-2}}{\|x-S'(\infty)\|^{d-2}}
\end{aligned}$$

Hence,

$$\begin{aligned}
\frac{1}{\|x-I_j(\infty)\|^{d-2}}h(I_j(x)) &= 1+\frac{1}{\|x-I_j(\infty)\|^{d-2}} + \sum_{S'\in\mathcal{G}-\{e,I_j\}} \frac{\alpha_{S'}^{d-2}}{\|x-S'(\infty)\|^{d-2}} \\
&= h(x)
\end{aligned}$$

by definition.

□

3.3 The Superharmonicity of the Function h

It remains to be shown that the function h we constructed in the previous section is indeed superharmonic. Notice that h is an increasing limit of superharmonic functions. We will prove that this limit is finite except on a polar set.

PROPOSITION 3.3.1. The function h is a superharmonic function.

PROOF. Recall the construction of the function h:

$$h(x) = 1 + \sum_{S\in\mathcal{G}-\{e\}} \frac{\alpha_S^{d-2}}{\|x-S(\infty)\|^{d-2}}, \quad \forall x\in\mathbb{R}^d - C.$$

Now, for $k\in\mathbb{N}$, let us define

$$h_k(x) = 1 + \sum_{S\in\mathcal{G}-\{e\},\, L(S)\le k} \frac{\alpha_S^{d-2}}{\|x-S(\infty)\|^{d-2}}, \quad \forall x\in\mathbb{R}^d - C.$$

Since the constant function and functions of the form

$$f(x) = \frac{1}{||x - b||^{d-2}}$$

are all harmonic functions in $\mathbb{R}^d$, it is easy to see that for any k, $h_k(x)$ is also a harmonic function. Therefore, it is superharmonic.

Notice that $h_{k+1}(x) > h_k(x)$ for all k and

$$\lim_{k \to \infty} h_k(x) = h(x).$$

Therefore, h is an increasing limit of superharmonic functions.

By the unique representation of S, the cardinality of the set

$$\mathcal{G}_k = \{S | S \in \mathcal{G}\,,\ L(S) = k\}$$

is $n(n-1)^{k-1}$ and we have

$$\mathcal{G} - \{e\} = \cup_{k=1}^{\infty} \mathcal{G}_k$$

Now let us examine each individual term of h.

Since for any $i \neq j$, $||c_i - c_j|| > n$, as we stipulated in chapter 2, we can estimate the magnitude of each α_S.

For any S, D_S is the product of $L(S) - 1$ factors, each of which is of the form

$$||I_j(\infty) - T(\infty)||$$

where $T \in \mathcal{G} - \{e\}$ and $P_1(T) = I_i \neq I_j$. Since

$$T(\infty) \in B_i$$

and

$$I_j(\infty) = c_j$$

it follows that

$$||I_j(\infty) - T(\infty)|| > ||c_i - c_j|| - 1 = n - 1$$

Furthermore, we have

$$\inf\{\|I_j(\infty) - T(\infty)\|, j = 1, 2, \dots, n;\ P_1(T) = I_i \neq I_j\} > n - 1$$

or, more explicitly, we have

$$\delta = \frac{1}{n-1} \inf\{\|I_j(\infty) - T(\infty)\|, j = 1, 2, \dots, n;\ P_1(T) = I_i \neq I_j\} > 1$$

So, for any S such that $L(S) = k$, $D_S > (n-1)^{k-1}\delta^{k-1}$. Consequently,

$$\alpha_S < \frac{1}{\delta^{k-1}(n-1)^{k-1}}$$

For any given $x \in E$, and any $S \in \mathcal{G} - \{e\}$,

$$\|x - S(\infty)\| \geq M$$

for some $M > 0$. Thus,

$$\frac{\alpha_S^{d-2}}{\|x - S(\infty)\|^{d-2}} < \frac{1}{(n-1)^{(k-1)(d-2)}\delta^{(k-1)(d-2)} M^{d-2}}$$

and

$$\begin{aligned} \sum_{S \in \mathcal{G} - \{e\},\, L(S)=k} \frac{\alpha_S^{d-2}}{\|x - S(\infty)\|^{d-2}} &< \frac{n(n-1)^{k-1}}{(n-1)^{(k-1)(d-2)}\delta^{(k-1)(d-2)} M^{d-2}} \\ &= \frac{n}{M^{d-2}(n-1)^{(k-1)(d-3)}} \left(\frac{1}{\delta^{d-2}}\right)^{k-1} \end{aligned}$$

Hence, the infinite series

$$h(x) = 1 + \sum_{k=1}^{\infty} \sum_{S \in \mathcal{G} - \{e\},\, L(S)=k} \frac{\alpha_S^{d-2}}{\|x - S(\infty)\|^{d-2}}$$

is convergent for all $x \in E$, i.e., $h(x)$ is finite in E.

Now we have proved that function h is a superharmonic function which is finite except on a polar set.

□

CHAPTER 4
MAIN RESULT

4.1 The Construction of X_t

In $\mathbb{R}^d$, $d \geq 3$, the inversion of Brownian motion, even in the simplest one-ball case, is no longer a Brownian motion, as is demonstrated in [15], p.100. It is an h-transform of Brownian motion for a superharmonic function h. The question is, what if we start with a process which is already an h-transform of Brownian motion? The answer is trivial in the one-ball case, due to the idempotent property of inversions.

Let I be the inversion with respect to the unit sphere, centered at the origin with radius 1. Suppose $Z_t = I(B_t)$ where B_t is Brownian motion. Z_t then is an h-transform of Brownian motion, where

$$h(x) = \frac{1}{||x||^{d-2}}$$

It is easy to see that $I(Z_t) = I^2(B_t) = B_t$. Notice that the group of homeomorphisms on $\mathbb{R}^d$ introduced by the inversion is finite, namely, $\mathcal{G} = \{e, I\}$.

However, in the case of two or more balls, the cardinality of this group becomes infinite, due to the fact that inversions with respect to different spheres do not commute. This means that the idempotent property of each individual inversion will no longer give us a clear indication as to what the Brownian motion will "become" after it has undergone a series of inversions.

Our solution here is to construct a diffusion process X which is invariant under any of these inversions. Specifically, by invariant we mean that if S is a member of $\mathcal{G}$, then for any $x \in \mathbb{R}^d - C$, (X_t, P^x) and $(S(X_t), P^{S^{-1}(x)})$ are identical in law.

We shall show that this particular diffusion can be obtained by an h-transform and a time change of Brownian motion.

Recall that

$$h(x) = 1 + \sum_{S \in \mathcal{G} - \{e\}} \frac{\alpha_S^{d-2}}{\|x - S(\infty)\|^{d-2}}, \quad \text{for all } x \in \mathbb{R}^d - C$$

is a positive superharmonic function and

$$\mathbb{R}^d - C = \{h < \infty\}$$

Let $B = (B_t, \mathcal{F}'_t)$ be the standard Brownian motion in $\mathbb{R}^d$ with transition density $p(t, x, y)$ and

$$B^h = (B_t^h, \mathcal{F}_t^h)$$

be the h-transform of B. The transition density of B^h is then

$$p^h(t, x, y) = \begin{cases} h(y)p(t, x, y)h(x)^{-1} & \text{if } (x, y) \in (\mathbb{R}^d - C) \times \mathbb{R}^d \\ 0 & \text{if } (x, y) \in C \times \mathbb{R}^d \end{cases}$$

Then B^h is a Markov process with state space $\mathbb{R}^d$, transition density $p^h(t, x, y)$, and initial distribution supported by $\mathbb{R}^d - C$. B^h also has an almost surely continuous modification and the strong Markov property.

Let $C_b(\mathbb{R}^d)$ be the class of bounded continuous functions on $\mathbb{R}^d$. If P_t is the semigroup of B and P_t^h is the semigroup of B^h, then for any $f \in C_b(\mathbb{R}^d)$, $t \geq 0$ and $x \in \mathbb{R}^d - C$,

$$P_t^h f(x) = \frac{1}{h(x)} P_t(hf)(x) \tag{4.1}$$

In $\mathbb{R}^d$, $d \geq 3$, it is easy to see that C is a polar set of B since it is countable. By Doob [12], p.674, almost no path of B^h hits C at a strictly positive time.

Next, we consider a time change of B^h.

Define

$$A_t = \int_0^t g(B_s^h)\, ds,$$

where g is the positive continuous function defined in section 3.1. Hence A_t defines a perfect continuous homogeneous additive functional (PCHAF) as in [18]. Let $\tau(t)$ be its right-continuous inverse, i.e.,

$$\tau(t) = \inf\{s : A_s > t\}$$

so

$$\{\tau(s) < t\} = \{A_t > s\}$$

Since $\{\mathcal{F}_t^h\}$ is right-continuous, for each s, $\tau(s)$ is an $\{\mathcal{F}_t^h\}$ stopping time. Notice that because

$$g(x) \geq 1$$

for all $x \in H^c$, we have

$$A_s \geq s$$

for all $s \geq 0$. Hence, for all $t \geq 0$,

$$\tau(t) \leq t \tag{4.2}$$

This is a useful condition in formulating the martingale problem later.

Set $X_t = B^h_{\tau(t)}$ and $\mathcal{F}_t = \mathcal{F}^h_{\tau(t)}$. Then $X = (X_t, \mathcal{F}_t)$ is a time change of B^h. This means that X is a strong Markov process just as B^h is. The time change of B^h preserves the continuity of its paths; so X also has continuous paths.

In conclusion, starting from a Brownian motion in $\mathbb{R}^d$, we have constructed a diffusion process X path-by-path in $\mathbb{R}^d - C$. In the next three sections, we shall characterize its distribution and then examine its invariance and symmetry properties.

4.2 Infinitesimal Generators

Having constructed the diffusion process X path-by-path, we can now examine its distribution. We can achieve this by characterizing its infinitesimal generator.

Let G be the infinitesimal generator of Brownian motion in $\mathbb{R}^d$ and $\mathcal{D}_G$ be its domain. Let $C_0(\mathbb{R}^d)$ be the space of bounded continuous functions on $\mathbb{R}^d$ which vanish at infinity. Let Δ be the Laplace operator. The natural domain of the operator $\frac{1}{2}\Delta$ is

$$\mathcal{D}(\frac{1}{2}\Delta) = \{f \in C_0(\mathbb{R}^d) : \frac{1}{2}\Delta f \text{ exists and } \frac{1}{2}\Delta f \in C_0(\mathbb{R}^d)\}$$

For $d \geq 2$, the operator G is the closure of $\frac{1}{2}\Delta$ in the following sense: $f \in \mathcal{D}_G$ if and only if there exist functions f_n in $\mathcal{D}(\frac{1}{2}\Delta)$ and a function g in $C_0(\mathbb{R}^d)$ such that $\|f_n - f\| \to 0$ and $\|\frac{1}{2}\Delta f_n - g\| \to 0$; and then $Gf = g$.

Let $C_0^2(\mathbb{R}^d - C)$ denote the class of twice continuously differentiable functions with compact support on $\mathbb{R}^d - C$. Then clearly

$$C_0^2(\mathbb{R}^d - C) \subset \mathcal{D}(\frac{1}{2}\Delta)$$

By the definition of h, it is twice continuously differentiable on $\mathbb{R}^d - C$. So, for any $f \in C_0^2(\mathbb{R}^d - C)$, we have

$$hf \in C_0^2(\mathbb{R}^d - C) \subset \mathcal{D}(\frac{1}{2}\Delta)$$

From this we can compute the infinitesimal generator of P_t^h, the semigroup of B_t^h. Recall equation (4.1) and let G^h denote the infinitesimal generator of P_t^h. Then

$$\begin{aligned} G^h f(x) &= \lim_{t \downarrow\downarrow 0} \frac{P_t^h f(x) - f(x)}{t} \\ &= \frac{1}{h(x)} \lim_{t \downarrow\downarrow 0} \frac{P_t(hf)(x) - h(x)f(x)}{t} \\ &= \frac{1}{2h(x)} \Delta(hf)(x) \end{aligned}$$

It is necessary to point out here that in $\mathbb{R}^d - C$, $d \geq 3$, the domain of G^h is richer than $C_0^2(\mathbb{R}^d - C)$. What has been established here is that P_t^h, the semigroup of B_t^h, has infinitesimal generator G^h where

$$G^h f = \frac{1}{2h} \Delta(hf)$$

for $f \in C_0^2(\mathbb{R}^d - C)$.

Next we turn to investigate the infinitesimal generator of the semigroup of X_t by studying the related martingale problem.

From the above discussion, we know that for any $f \in C_0^2(\mathbb{R}^d - C)$,

$$C_t^f \equiv f(B_t^h) - \int_0^t G^h f(B_s^h) ds$$

is a P^x-martingale for any $x \in \mathbb{R}^d - C$. By the definition of $\tau(t)$ and equation (4.2), we know that $\tau(t)$ is an optional stopping time with respect to $\mathcal{F}_t^h$ and

$$C^f_{\tau(t)} = f(B^h_{\tau(t)}) - \int_0^{\tau(t)} G^h f(B^h_s)ds$$

is also a P^x-martingale for any $x \in \mathbb{R}^d - C$ with respect to $\mathcal{F}_t^h$.

Letting $s = \tau(u)$, we have

$$u = A_s = \int_0^s g(B^h_v)dv$$

It then follows that $du = g(B^h_s)ds$ and

$$\begin{aligned} C^f_{\tau(t)} &= f(B^h_{\tau(t)}) - \int_0^t G^h f(B^h_{\tau(u)}) \frac{1}{g(B^h_{\tau(u)})} du \\ &= f(X_t) - \int_0^t g^{-1}(X_u) G^h f(X_u) du \end{aligned}$$

Since $\tau(0) = 0$,

$$E^x C^f_{\tau(t)} = E^x C^f_{\tau(0)} = E^x C^f_0 = E^x f(B^h_0) = f(x)$$

and

$$\begin{aligned} f(x) &= E^x f(X_t) - \int_0^t E^x(g^{-1}(X_u)G^h f(X_u))du \\ &= \tilde{P}_t f(x) - \int_0^t \tilde{P}_u(g^{-1}G^h f)(x)du \end{aligned}$$

where $\tilde{P}_t$ is the semigroup of X_t.

Now we have

$$\tilde{P}_t f(x) - f(x) = \int_0^t \tilde{P}_u(g^{-1}G^h f)(x)du$$

and if we denote the infinitesimal generator of $\tilde{P}_t$ by $\mathcal{A}$,

$$\begin{aligned} \mathcal{A}f(x) &= \lim_{t \downarrow\downarrow 0} \frac{\tilde{P}_t f(x) - f(x)}{t} \\ &= (g^{-1}G^h f)(x) \\ &= \frac{1}{g(x)} G^h f(x) \\ &= \frac{1}{2g(x)h(x)} \Delta(hf)(x) \end{aligned}$$

Thus, we have established that the infinitesimal generator of $\tilde{P}_t$, the semigroup of X, acting on $C_0^2(\mathbb{R}^d - C)$ is

$$\mathcal{A}f = \frac{1}{2gh}\Delta(hf)$$

Again, $C_0^2(\mathbb{R}^d - C)$ is only a subspace of the domain of $\mathcal{A}$. The latter is in fact richer.

4.3 Invariance of X_t and The Horizon H

In this section, we shall show that the process X_t is invariant in law under inversion. Based on this, we can further prove that the horizon set H is a polar set for X_t.

THEOREM 4.3.1. For any $S \in \mathcal{G}$, the process $(S(X_t), P^{S^{-1}(x)})$ and the process (X_t, P^x) are identical in law.

PROOF. Our proof is based on the fact that if the potential of $(S(X_t), P^{S^{-1}(x)})$ is the same as that of (X_t, P^x), then they are identical in law.

It is sufficient to prove the case of $S = I_j$ for any $1 \leq j \leq n$. In the following proof, let E_x denote the expectation with respect to the distribution of X_t starting at point x and let E_x^h denote that of B_t^h starting at point x.

$$\begin{aligned}
& E_{I_j(x)} \int_0^\infty f(I_j(X_t))\,dt \\
= \;& E^h_{I_j(x)} \int_0^\infty f(I_j(B^h_{\tau_t}))\,dt \\
= \;& E^h_{I_j(x)} \int_0^\infty f(I_j(B^h_s))g(B^h_s)\,ds \\
= \;& \frac{1}{h(I_j(x))} \int_{\mathbb{R}^d} \int_0^\infty p_t(I_j(x),\, y) f(I_j(y)) g(y) h(y)\,ds\,dy
\end{aligned}$$

where

$$p_t(I_j(x),\, y) = \frac{1}{(2\pi t)^{d/2}} e^{-\frac{\|y - I_j(x)\|^2}{2t}}$$

Through an elementary calculation, if we denote $\frac{1}{2}\Gamma(\frac{d}{2}-1)\pi^{-\frac{d}{2}}$ by K, we have

$$\int_0^\infty p_t(I_j(x),\, y)\, dt = K\|y - I_j(x)\|^{2-d}$$

Therefore, if we let $z = I_j(y)$, $y = I_j(z)$, and recall that $\left|\frac{\partial y}{\partial z}\right| = \|z - c_j\|^{-2d}$, we have

$$\begin{aligned}
& E_{I_j(x)} \int_0^\infty f(I_j(X_t))\, dt \\
= \; & \frac{K}{h(I_j(x))} \int_{\mathbb{R}^d} f(I_j(y)) g(y) h(y) \|y - I_j(x)\|^{2-d}\, dy \\
= \; & \frac{K}{h(I_j(x))} \int_{\mathbb{R}^d} f(z) g(I_j(z)) h(I_j(z)) \|I_j(z) - I_j(x)\|^{2-d} \|z - c_j\|^{-2d}\, dz
\end{aligned}$$

Recall that

$$g(I_j(z)) = \|z - c_j\|^4 g(z)$$

by Proposition 3.1.2 and

$$h(I_j(z)) = \|z - c_j\|^{d-2} h(z)$$

by Proposition 3.2.2. Furthermore, we have

$$\|I_j(x) - I_j(y)\| = \frac{\|x - y\|}{\|x - c_j\| \times \|y - c_j\|}$$

according to Proposition 2.1.2. Therefore,

$$\begin{aligned}
& E_{I_j(x)} \int_0^\infty f(I_j(X_t))\, dt \\
= \; & \frac{K}{h(x)\|x - c_j\|^{d-2}} \times \\
& \int_{\mathbb{R}^d} f(z) g(z) \|z - c_j\|^4 h(z) \|z - c_j\|^{d-2} \frac{\|z - x\|^{2-d}}{\|z - c_j\|^{2-d} \|x - c_j\|^{2-d}} \|z - c_j\|^{-2d}\, dz
\end{aligned}$$

After simplifying the right hand side, we get

$$E_{I_j(x)} \int_0^\infty f(I_j(X_t))\, dt = \frac{K}{h(x)} \int_{\mathbb{R}^d} f(z) g(z) h(z) \|z - x\|^{2-d}\, dz$$

On the other hand,

$$\begin{aligned}
& E_x \int_0^\infty f(X_t)\,dt \\
= \; & E_x^h \int_0^\infty f(B^h_{\tau_t})\,dt \\
= \; & E_x^h \int_0^\infty f(B^h_s) g(B^h_s)\,ds \\
= \; & \frac{1}{h(x)} \int_{\mathbb{R}^d} \int_0^\infty p_t(x,\, y) f(y) g(y) h(y)\,ds\,dy \\
= \; & \frac{K}{h(x)} \int_{\mathbb{R}^d} f(y) g(y) h(y) \|y - x\|^{2-d}\,dy
\end{aligned}$$

Thus, these two processes are identical in law by Blumenthal and Getoor [7].

□

Now consider the horizon set H. As shown in section 2.3,

$$H = \cap_{S \in \mathcal{G}} S(\cup_{j=1}^n B_j)$$

For any $S \in \mathcal{G} - \{e\}$, let $t(S)$ be the index of the rightmost inversion in the unique representation of S, i.e., for

$$S = I_{n_1} I_{n_2} \ldots I_{n_k}$$

let $t(S) = n_k$.

Then, for any $n > 0$, let

$$H_n = \cup_{S \in \mathcal{G}_n} S(\cup_{j \neq t(S)} B_j),$$

where $\mathcal{G}_n = \{S : S \in \mathcal{G}, L(S) = n\}$. It is clear that $H \subset H_{n+1} \subset H_n$ for all n.

Let

$$T = \inf\{t : X_t \in H\}$$

and

$$T_n = \inf\{t : X_t \in H_n\},$$

then, since $H_{n+1} \subset H_n$,

$$\{T_{n+1} < \infty\} \subset \{T_n < \infty\}$$

and

$$\{T < \infty\} = \cap_{n=1}^{\infty} \{T_n < \infty\}$$

We need the following lemma to prove that H is a polar set for X_t.

LEMMA 4.3.1. For any $x \in \cup_{j=1}^{n} \partial B_j$,

$$P^x(T_2 < \infty) \leq m < 1$$

PROOF. First, since X_t is transient and H_2 is compact,

$$P^x(T_2 < \infty) < 1$$

for any $x \in \cup_{j=1}^{n} \partial B_j$.

Since $x \mapsto P^x(T_2 < \infty)$ is harmonic on the interior of H_2^c, it is continuous on the compact set $\cup_{j=1}^{n} \partial B_j$. Therefore, by a standard argument in calculus,

$$m = \max_{x \in \cup_{j=1}^{n} \partial B_j} P^x(T_2 < \infty) < 1.$$

□

Now we are ready to prove the following theorem.

THEOREM 4.3.2. For any $x \in \mathbb{R}^d - C$,

$$P^x(T < \infty) = 0$$

PROOF. For any $n \geq 1$, $x \in \mathbb{R}^d - C$,

$$\begin{aligned} P^x(T_{n+1} < \infty) &= P^x(T_n + T_{H_{n+1}} \circ \theta_{T_n} < \infty) \\ &= P^x(T_n < \infty, P^{X_{T_n}}(T_{H_{n+1}} < \infty)) \\ &= P^x(P^{X_{T_n}}(T_{H_{n+1}} < \infty) P^x(T_n < \infty)) \end{aligned}$$

For every $\omega \in \Omega$, $X_{T_n}(\omega) \in S(\cup_{j=1}^n \partial B_j)$ for some $S \in \mathcal{G}_{n-1}$. Hence, by Theorem 4.3.1,

$$\begin{aligned} P^{X_{T_n}}(T_{H_{n+1}} < \infty) &= P^{X_{T_n}}(X_t \in H_{n+1} \text{ for some } t < \infty) \\ &= P^{S^{-1}(X_{T_n})}(S(X_t) \in H_{n+1} \text{ for some } t < \infty) \\ &= P^{S^{-1}(X_{T_n})}(X_t \in S^{-1}(H_{n+1}) \text{ for some } t < \infty) \\ &\leq P^{S^{-1}(X_{T_n})}(X_t \in H_2 \text{ for some } t < \infty) \end{aligned}$$

Since $S^{-1}(X_{T_n}) \in \cup_{j=1}^n \partial B_j$, by Lemma 4.3.1,

$$P^{X_{T_n}}(T_{H_{n+1}} < \infty) \leq m < 1.$$

Thus,

$$\begin{aligned} P^x(T_{n+1} < \infty) &\leq mP^x(P^x(T_n < \infty)) \\ &= mP^x(T_n < \infty) \end{aligned}$$

By induction on n, we have

$$P^x(T_{n+1} < \infty) \leq m^n \text{ for all } n \geq 1.$$

Therefore,

$$\begin{aligned} P^x(T < \infty) &= \lim_{n\to\infty} P^x(T_{n+1} < \infty) \\ &\leq \lim_{n\to\infty} m^n \\ &= 0 \end{aligned}$$

for all $x \in \mathbb{R}^d - C$.

□

A set theory analysis shows that $|H| = 2$ for the case $n = 2$. For $n > 2$, H is a Cantor-like set.

For $n = 2$,

$$H = H_1 \cup H_2,$$

where

$$H_1 = \cap_{n=0}^{\infty} (I_1 I_2)^n I_1(\overline{B_2}),$$

and

$$H_2 = \cap_{n=0}^{\infty} (I_2 I_1)^n I_2(\overline{B_1}).$$

Both H_1 and H_2 are intersections of nested closed sets in $\mathbb{R}^d$. Since the radii of $(I_1 I_2)^n I_1(\overline{B_2})$ and $(I_2 I_1)^n I_2(\overline{B_1})$ approach zero as $n \to \infty$, each of H_1 and H_2 contains exactly one point, respectively. So, $|H| = 2$.

For $n \geq 3$, rewrite H as

$$H = \cap_{n=1}^{\infty} \cup_{L(S)=n} S(\cup_{j \neq t(S)} \overline{B_j})$$

Then we can see that H is the intersection of a sequence of nested closed sets. Therefore H is closed.

4.4 Symmetry of X_t

In this section, we briefly summarize some important facts about dual processes and symmetric processes relevant to our study. It then follows that the strong Markov process X_t is a symmetric process.

Suppose (Ω, Z_t, P_t) is a Markov process in $\mathbb{R}^d$ and U is its potential operator. Then, for any $f \in C^\infty(\mathbb{R}^d)$, we have

$$Uf(x) = \int_0^\infty P_t f(x)\, dt.$$

Suppose the semigroup operator P_t has a density function $p_t(x, y)$ with respect to a measure ζ, then it is easy to see that U has a kernel $u(x, y)$ with respect to ζ and

$$Uf(x) = \int_{\mathbb{R}^d} f(y)u(x, y)\zeta(dy),$$

where

$$u(x, y) = \int_0^\infty p_t(x, y)\, dt.$$

The *dual* process of (Ω, Z_t, P_t) with respect to η is defined to be a Markov process $(\Omega, \hat{Z}_t, \hat{P}_t)$ such that

$$\int_{\mathbb{R}^d} P_t f(x)g(x)\eta(dx) = \int_{\mathbb{R}^d} f(x)\hat{P}_t g(x)\eta(dx)$$

If $\hat{U}$ is the potential operator of $(\Omega, \hat{Z}_t, \hat{P}_t)$ with kernel $\hat{u}(x, y) = u(y, x)$, then we have

$$\int_{\mathbb{R}^d} (Uf(x))g(x)\eta(dx) = \int_{\mathbb{R}^d} f(x)\hat{U}(g(x))\eta(dx).$$

If $\hat{U} = U$, we say Markov process (Ω, Z_t, P_t) is *symmetric* with respect to η.

Next, let us consider the h-transform of (Ω, Z_t, P_t), for some positive superharmonic function h. If we denote this process as (Ω, Z_t^h, P_t^h) and let $(\Omega, \hat{Z}_t, \hat{P}_t)$ be the dual process of (Ω, Z_t, P_t) with respect to reference measure ζ, then we have the following fact.

PROPOSITION 4.4.1. Suppose (Ω, Z_t, P_t) is a Markov process and P_t has density function $p_t(x, y)$ with respect to ζ and $(\Omega, \hat{Z}_t, \hat{P}_t)$ is its dual process with respect to a reference measure ζ. Then for any positive superharmonic function h, the Markov processes (Ω, Z_t^h, P_t^h) and $(\Omega, \hat{Z}_t, \hat{P}_t)$ are dual processes with respect to the reference measure $h\zeta$.

PROOF. We know

$$p_t^h(x, y) = \frac{h(y)}{h(x)} p_t(x, y).$$

So, for any $f,\, g \in\ C^\infty(\mathbb{R}^d)$,

$$\begin{aligned}
&\int_{\mathbb{R}^d} (P_t^h f(x)) g(x) h(x) \zeta(dx) \\
=\ & \int_{\mathbb{R}^d} [\int_{\mathbb{R}^d} f(y) p_t^h(x,y) \zeta(dy)] g(x) h(x) \zeta(dx) \\
=\ & \int_{\mathbb{R}^d} [\int_{\mathbb{R}^d} f(y) \frac{h(y)}{h(x)} p_t(x,y) \zeta(dy)] g(x) h(x) \zeta(dx) \\
=\ & \int_{\mathbb{R}^d} [\int_{\mathbb{R}^d} f(y) h(y) p_t(x,y) \zeta(dy)] g(x) \zeta(dx) \\
=\ & \int_{\mathbb{R}^d} f(y) h(y) [\int_{\mathbb{R}^d} p_t(x,y) g(x) \zeta(dx)] \zeta(dy) \\
=\ & \int_{\mathbb{R}^d} f(y) [\int_{\mathbb{R}^d} \hat{p}_t(y,x) g(x) \zeta(dx)] h(y) \zeta(dy) \\
=\ & \int_{\mathbb{R}^d} f(y) (\hat{P}_t g(y)) h(y) \zeta(dy)
\end{aligned}$$

Hence, (Ω, Z_t^h, P_t^h) and $(\Omega, \hat{Z}_t, \hat{P}_t)$ are dual to each other with respect to $h\zeta$.

□

Now that we have established that (Ω, Z_t^h, P_t^h) and $(\Omega, \hat{Z}_t, \hat{P}_t)$ are dual processes, we may use this result once more to conclude the following result.

COROLLARY 4.4.1. The Markov processes $(\Omega, \hat{Z}_t^h, \hat{P}_t^h)$ and (Ω, Z_t^h, P_t^h) are dual processes with respect to $h^2\zeta$.

PROOF. By Proposition 4.4.1, $(\Omega, \hat{Z}_t, \hat{P}_t)$ is the dual process of (Ω, Z_t^h, P_t^h) with respect to $h\zeta$. So, since the h-transform of the former is $(\Omega, \hat{Z}_t^h, \hat{P}_t^h)$, using Proposition 4.4.1 again, we conclude that this process is the dual process of (Ω, Z_t^h, P_t^h) with respect to $h^2\zeta$.

□

Now we are ready to state the central result of this section.

THEOREM 4.4.1. If a Markov process (Ω, Z_t, P_t) is symmetric, then so is its h-transform (Ω, Z_t^h, P_t^h).

PROOF. Suppose (Ω, Z_t, P_t) is symmetric with respect to ζ. Then $(\Omega, \hat{Z}_t, \hat{P}_t) = (\Omega, Z_t, P_t)$. So, $(\Omega, \hat{Z}_t^h, \hat{P}_t^h) = (\Omega, Z_t^h, P_t^h)$. By definition, (Ω, Z_t^h, P_t^h) is symmetric with respect to $h^2\zeta$.

□

From this, we have the following.

PROPOSITION 4.4.2. The Markov process X_t is symmetric.

PROOF. In fact, from the discussion above, we can identify the potential V of (Ω, B_t^h, P_t^h): for any $f \in C^\infty(\mathbb{R}^d)$ and $x \in H^c$,

$$Vf(x) = \int_{\mathbb{R}^d} f(y)v(x,y)\eta(dy)$$

where $\eta = h^2\lambda$ and

$$v(x,y) = \int_0^\infty \frac{p(t,x,y)}{h(x)h(y)}dt.$$

Let U denote the potential of X_t. Let E_x and E_x^h denote expectations with respect to the distribution of X_t and that of B_t^h, respectively. Then for any $f \in C^\infty(\mathbb{R}^d)$ and $x \in H^c$,

$$\begin{aligned} Uf(x) &= E_x \int_0^\infty f(X_t)dt \\ &= E_x^h \int_0^\infty f(B_{\tau(t)}^h)dt \\ &= E_x^h \int_0^\infty f(B_s^h)g(B_s^h)ds \\ &= V(gf)(x) \\ &= \int_{\mathbb{R}^d} f(y)v(x,y)g(y)\eta(dy) \\ &= \int_{\mathbb{R}^d} f(y)v(x,y)\xi(dy) \end{aligned}$$

where $\xi = g\eta$ and $v(x,y)$ is symmetric as shown above. Hence, X_t is symmetric.

□

By now, we have proved that X_t is a symmetric diffusion process which is invariant in law under any $S \in \mathcal{G}$.

4.5 The Reflected Diffusion W_t

Now we are ready to identify the reflected diffusion process in E. Recall that Φ is continuous from $\mathbb{R}^d$ onto E. It has the property

$$\Phi \circ S = \Phi$$

for every $S \in \mathcal{G}$. This means that Φ is constant on any orbit $[x]$.

LEMMA 4.5.1. For any continuous function f on E, $\tilde{P}_t(f \circ \Phi)(x)$ is constant on the orbit $[x]$.

PROOF. By Theorem 4.3.1, the processes (X_t, P^x) and $(S(X_t), P^{S^{-1}(x)})$ are identical in law for any $S \in \mathcal{G}$. Thus, for any continuous function f on E,

$$P^x[(f \circ \Phi)(X_t)] = P^{S^{-1}x}[(f \circ \Phi)(S(X_t))]$$

But since $\Phi \circ S = \Phi$, $f \circ \Phi \circ S = f \circ \Phi$. We have

$$P^x[(f \circ \Phi)(X_t)] = P^{S^{-1}x}[(f \circ \Phi)(X_t)]$$

Therefore,

$$\tilde{P}_t(f \circ \Phi)(x) = \tilde{P}_t(f \circ \Phi)(S^{-1}(x))$$

for any $S \in \mathcal{G}$. This proves that $\tilde{P}_t(f \circ \Phi)(x)$ is constant on the orbit $[x]$.

☐

COROLLARY 4.5.2. Let $\{U^\alpha; \alpha > 0\}$ be the resolvent of the semigroup $\{\tilde{P}_t; t \geq 0\}$. Then for every continuous function f on E,

$$U^\alpha(f \circ \Phi)(x) = E^x \int_0^\infty e^{-\alpha t}(f \circ \Phi)(X_t)dt$$

is constant on $[x]$.

Thanks to these facts, we can define a Markov transition function on $(E, \mathcal{B}(E))$.

DEFINITION 4.5.3. For any $t \in [0,\infty)$, $x \in E$ and $A \in \mathcal{B}(E)$, let

$$Q_t(x,A) = \tilde{P}_t^y(X_t \in \Phi^{-1}(A))$$

where $y \in [x]$.

By Lemma 4.5.1, $Q_t(x,A)$ is well defined since $\tilde{P}_t(f \circ \Phi)(x)$ is constant on the orbit $[x]$. The next theorem shows that it is a Markov transition function.

THEOREM 4.5.4. The collection $\{Q_t(\cdot,\cdot), t \in [0,\infty)\}$ is a Markov transition function on $(E, \mathcal{B}(E))$.

PROOF. First, since X_t is a Markov process, for any $x \in E$ and $A \in \mathcal{B}(E)$, $A \to Q_t(x,A)$ is a probability measure and $x \to Q_t(x,A)$ is $\mathcal{B}(E)$-measurable.

Second, if we write

$$Q_t f(x) = \int_E f(z) Q_t(x,dz)$$

then

$$\begin{aligned} Q_t(x,A) &= (Q_t I_A)(x) \\ &= (\tilde{P}_t I_{\Phi^{-1}(A)})(y) \\ &= \tilde{P}_t(I_A \circ \Phi)(y) \end{aligned}$$

for any $y \in [x]$.

By a standard measure theory argument, for any bounded continuous function f on E, we have

$$Q_t f(x) = \tilde{P}_t(f \circ \Phi)(y)$$

for any $y \in [x]$.

Therefore, for any $s > 0$,

$$\begin{aligned} Q_{t+s} f(x) &= \tilde{P}_{t+s}(f \circ \Phi)(y) \\ &= \tilde{P}_t(\tilde{P}_s(f \circ \Phi))(y) \end{aligned}$$

Since $\tilde{P}_s(f \circ \Phi)(z) = Q_s f(x)$ for $x = \Phi(z)$, we have

$$\tilde{P}_s(f \circ \Phi)(z) = ((Q_s f) \circ \Phi)(z)$$

therefore,

$$\begin{aligned} Q_{t+s} f(x) &= \tilde{P}_t((Q_s f) \circ \Phi)(y) \\ &= Q_t(Q_s f)(x) \end{aligned}$$

Thus, $\{Q_t(\cdot, \cdot), t \in [0, \infty)\}$ is a Markov transition function.

☐

We can also define a resolvent on E by setting

$$V^\alpha f(x) = U^\alpha(f \circ \Phi)(y)$$

where $y \in [x]$.

For every ω in the underlying probability space $(\Omega, \mathcal{F}, P)$, define

$$W_t(\omega) = \Phi(X_t(\omega)) \text{ for all } t \geq 0.$$

Notice here that a path of W_t is derived from the path of X_t. It is easy to see by the definition of Φ that for every fixed ω the path $\Phi(X_t(\omega))$ lies in E.

The next theorem shows that W_t is a Markov process in E with Q_t as its transition function.

THEOREM 4.5.5. W_t is a strong Markov process in E with continuous paths.

PROOF. First, by the above construction, all paths of W_t lie in E and they are all continuous.

For every bounded continuous function f on E and every $x \in E$,

$$\begin{aligned} Q_t f(x) &= \tilde{P}_t(f \circ \Phi)(x) \\ &= \tilde{E}^x(f \circ \Phi)(X_t) \\ &= E^x f(W_t) \end{aligned}$$

where $E^x(\cdot)$ is the expectation with respect to the distribution of W_t starting at x.

Moreover, for any bounded continuous function f, the map $t \to V^\alpha f(W_t)$ is right continuous since

$$V^\alpha f(W_t) = U^\alpha(f \circ \Phi)(X_t).$$

By the well known theorem of Blumenthal and Getoor, ([7], Chapter 1, Theorem (8.11)), W_t is a strong Markov process.

□

By the definition of Q_t, the Markov process W_t also inherits other properties of X_t.

THEOREM 4.5.6. W_t is a symmetric process.

PROOF. If $\hat{P}_t$ is the dual semigroup of $\tilde{P}_t$, then $\hat{P}_t(f \circ \Phi)(x)$ is also constant on $[x]$. So we can define $\hat{Q}_t$ similarly, and it is the dual semigroup of Q_t. Since $\hat{P}_t = \tilde{P}_t$ by the symmetry of X_t, we have

$$\hat{Q}_t = Q_t.$$

Therefore, W_t is a symmetric process.

□

THEOREM 4.5.7. W_t is a transient Markov process with infinite lifetime.

PROOF. Let $K \subset E$ be a compact set such that

$$\lambda(K) > 0$$

where λ is the Lebesgue measure. Then K is bounded and closed. By the definition of Φ, $\Phi^{-1}(K)$ is also bounded and closed. Therefore, $\Phi^{-1}(K)$ is compact in $\mathbb{R}^d$ and

$$\lambda(\Phi^{-1}(K) > 0).$$

By the transience property of X_t, for any $x \in E$,

$$0 < UI_{\Phi^{-1}(K)}(y) < \infty$$

for every $y \in [x]$. Since

$$VI_K(x) = UI_K \circ \Phi(y) = UI_{\Phi^{-1}(K)}(y),$$

it then follows that

$$0 < VI_K(x) < \infty.$$

Thus, W_t is also transient.

The fact that h is a polar set for X_t implies that X_t has infinite lifetime in H^c. Hence, the lifetime of W_t in E is also infinite. □

Therefore, we have demonstrated that W_t is a symmetric, transient diffusion process with infinite lifetime on E.

THEOREM 4.5.8. W_t is an h-Brownian motion in the interior of E and is normally reflected on the boundary

$$\partial E = \cup_{j=1}^{n} \partial B_j$$

PROOF. Let $C_0^2(E)$ be the class of twice continuously differentiable functions with compact support on E. Let $\mathcal{B}$ be the infinitesimal generator of Q_t. Then, for every $f \in C_0^2(E)$, $x \in E$,

$$\mathcal{B}f(x) = \lim_{t \downarrow\downarrow 0} \frac{Q_t f(x) - f(x)}{t}$$

and

$$f(W_t) - \int_0^t (\mathcal{B}f)(W_s) ds$$

is a P^x-martingale for every $x \in E$. Since

$$W_t(\omega) = \Phi(X_t(\omega)) \text{ for all } t \geq 0.$$

we know that

$$f \circ \Phi(X_t) - \int_0^t (\mathcal{B}f) \circ \Phi(X_s) ds$$

is a $\tilde{P}^x$-martingale. Hence, we have

$$\tilde{P}_t(f \circ \Phi)(x) - f \circ \Phi(x) = \int_0^t \tilde{P}_s((\mathcal{B}f) \circ \Phi)(x) ds$$

Therefore,

$$\begin{aligned}(\mathcal{B}f) \circ \Phi(x) &= \lim_{t \downarrow\downarrow 0} \frac{1}{t} \int_0^t \tilde{P}_s((\mathcal{B}f) \circ \Phi)(x) ds \\ &= \lim_{t \downarrow\downarrow 0} \frac{1}{t} (\tilde{P}_t(f \circ \Phi)(x) - f \circ \Phi(x)) \\ &= \mathcal{A}(f \circ \Phi)(x)\end{aligned}$$

For any $x \in E^o$, there exists a neighborhood of x, $N(x)$, such that $N(x) \subset E^o$. Therefore, we have

$$\Phi(z) = z$$

for all $z \in N(x)$. Thus, since $g(z) = 1$ for $z \in E$, it follows that

$$\mathcal{B}f(x) = \mathcal{A}f(x) = \frac{1}{2h(x)} \Delta(hf)(x)$$

This shows that W_t is an h-Brownian motion in E^o.

If $x \in \partial E = \cup_{j=1}^n \partial B_j$, then there exists a neighborhood of x, $N(x)$, such that $N(x) \subset B_j \cap E$ for some j. For any $z \in N(x) \cap E$,

$$\Phi(z) = z$$

and for any $z \in N(x) \cap B_j$,

$$\Phi(z) = I_j(z)$$

Let the center of B_j be $c_j = (c_j^1, c_j^2, \dots c_j^d)$. Let $f(z) = f(z_1, z_2, \dots, z_d)$ and

$$\nabla f = (\frac{\partial f}{\partial z_1}, \frac{\partial f}{\partial z_2}, \dots, \frac{\partial f}{\partial z_d})$$

Compute $\nabla(h(f \circ I_j))$, we get

$$\begin{aligned}\nabla(h(f \circ I_j))(z) &= (\nabla h)(z)(f \circ I_j)(z) + h(z) \nabla(f \circ I_j)(z) \\ &= (\nabla h)(z) f(I_j(z)) + h(z)(\nabla f)(I_j(z)) \left[\frac{1}{\|z - c_j\|^2} I - \frac{2}{\|z - c_j\|^4} D(z, c_j)\right]\end{aligned}$$

where I is the $d \times d$ identity matrix and

$$D(z, c_j) = \begin{pmatrix} z_1 - c_j^1 \\ z_2 - c_j^2 \\ \vdots \\ z_d - c_j^d \end{pmatrix} (z_1 - c_j^1, z_2 - c_j^2, \dots, z_d - c_j^d)$$

Compare this with

$$\nabla(hf)(z) = \nabla h(z) f(z) + h(z) \nabla f(z)$$

Notice that $x \in \partial B_j$ implies $I_j(x) = x$, $\|x - c_j\| = 1$, and with $g(x) = 1$, we have

$$\mathcal{B} f(x) = -\frac{1}{h(x)} (\nabla f)(x) D(x, c_j)$$

for every $x \in \partial B_j$, $j = 1, 2, \dots, n$.

Notice that for every j and any vector $\mathbf{v} = (v_1, v_2, \dots, v_d)$,

$$\mathbf{v} D(x, c_j) = K(x_1 - c_j^1, x_2 - c_j^2, \dots, x_d - c_j^d)$$

for some scalar K. So $\mathbf{v} D(x, c_j)$ is normal to the surface ∂B_j.

From the symmetry property of W_t and the fact that $\mathcal{B}$ is normal to ∂E on every boundary point, we know that W_t is an h-Brownian motion with normal reflection on the boundary.

$\square$

This concludes the construction.

CHAPTER 5
CONCLUSION

5.1 Summary

In the study of multidimensional diffusion processes, there are generally two approaches, as discussed in Stroock and Varadhan [8]. The analytic approach is one of them. In this approach, a diffusion process is studied through the parabolic equation associated with it, such as in the study by Fukushima [9]. The major difficulty with this approach is that it is generally a non-trivial problem to prove that this parabolic equation has a sufficiently large family of solutions with required properties to characterize the associated diffusion process. And since we can only learn about this process by studying the associated parabolic equation, probabilistic intuitions are somewhat obscured by analytic techniques.

Another approach is to study diffusion processes using Itô's stochastic integration. Examples along this line can be found in Ikeda and Watanabe [10] and Watanabe [11]. Since the diffusion processes are studied in association with the underlying Brownian motion, more probabilistic insight can be gained. In particular, a process can be constructed path-by-path, where the state space of the process and the underlying probability space are explicit.

Glover and Rao gave a path-by-path construction of reflected Brownian motion in $\mathbb{R}^2$ (See Glover and Rao [5]). They achieved this by geometrically reflecting the paths of a standard Brownian motion on the boundary of an open set and showing that the resulting process is a Brownian motion with a time change.

In this thesis, a reflected h-Brownian motion in $\mathbb{R}^d, d \geq 3$ is constructed path-by-path. The definition, properties of h-Brownian motion and its association with Brownian motion are discussed in Doob [12].

This path-by-path construction starts with a standard Brownian motion in $\mathbb{R}^d, d \geq 3$. An h-Brownian motion is then derived from it for a superharmonic function h. This h-Brownian motion is then time-changed to a diffusion process X_t. Under the conditions imposed on the superharmonic function h and the time change, X_t is a symmetric Markov process with infinite lifetime. A continuous many-to-one map $\Phi : \mathbb{R}^d \to E$ is defined, where $E = \mathbb{R}^d - \cup_{j=1}^n B_j$.

It is then proved that $W_t = \Phi(X_t)$ is a strong Markov process with state space E. W_t is also symmetric and has infinite lifetime. The infinitesimal generator of this process is then computed and it shows that W_t behaves like h-Brownian motion in the interior of E and has normal reflection on the boundary.

It is worth noticing that a many-to-one function in general does not map a Markov process to a Markov process. A study in this subject can be found in Glover and Mitro [6].

5.2 Discussion

The interest in studying reflected h-Brownian motion originated in the study of reflected Brownian motion and Markov processes in general. The connections between Markov processes and martingales, probability theory and potential theory, provide powerful tools in the study of these subjects. References on these subjects can be found in Chung [13], Rao [14], Durrett [15], Freedman [16], Revuz [17], Williams [18], McKean [19], Itô and McKean [20] and Landkof [21].

As pointed out in Chung [13], an analytical approach to the reflecting boundary problem in multidimensional spaces presents a major difficulty. In the case studied here, the construction is realized path-by-path.

There are three restrictions imposed on this construction, namely, the number of open balls B_j, their radii, and the minimum distance between them.

The restriction that each ball has radius 1 is for the sake of clarity only and can be removed without altering the the basic argument. That is, this construction will not change fundamentally if the balls are allowed to have different radii, but the symbols and formulas will be more complicated.

The number of balls and the minimum distance between them, however, present a significant difficulty in the construction of the superharmonic function h. Can a superharmonic function h be found that satisfies

$$h(x) = \frac{1}{\|x - I_j(\infty)\|^{d-2}} h(I_j(x))$$

for all $x \in \mathbb{R}^d - C, j = 1, 2, 3, \ldots$? More specifically, can we find a superharmonic function which satisfies the above property without resorting to an infinite sum as we have to do in this study? It is unclear even in the two ball case.

Another aspect concerning the distance between balls is the structure of the horizon set H, from a pure set theory point of view. This can probably be better illustrated in $\mathbb{R}^2$. If three non-overlapping open discs B_1, B_2 and B_3 all have positive distance between them, the set H, as the result of iterated inversions, is the intersection of a sequence of nested closed sets. It is a subset of a circle B intersecting B_1, B_2 and B_3. The linear Lebesgue measure of H, on B, is zero. However, if each pair of ∂B_1, ∂B_2 and ∂B_3 intersect at one point, i.e., if these three open discs are "touching" one another, then the resultant H is the entire circle passing through the three points

$$\partial B_1 \cap \partial B_2, \ \partial B_1 \cap \partial B_3 \ \text{ and } \ \partial B_2 \cap \partial B_3.$$

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BIOGRAPHICAL SKETCH

Xibing Wang was born in Tianjin, China, on October 4, 1961. He received a bachelor's degree in mathematics from Peking University in 1984. In July, 1987, he received a Master of Science degree in mathematics from Nankai University and worked as an assistant professor in its Department of Mathematics for a year and a half. At the end of 1988, he came to the United States and started his graduate study towards a doctoral degree at the University of Florida. In May, 1997, he received a Ph.D. degree in mathematics from the University of Florida, under the direction of Dr. Joseph Glover.

I certify that I have read this study and that in my opinion it conforms to acceptable standards of scholarly presentation and is fully adequate, in scope and quality, as a dissertation for the degree of Doctor of Philosophy.

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